MTHSC 3110 Section 1.9 – The Matrix of a Linear Transformation

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Recall

A function $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is called a linear transformation if for every $\vec{u}, \vec{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$,

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$.
2. $T(c\vec{u}) = cT(\vec{u})$

Example

Suppose $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is a linear transformation, and that we know

$$T(\vec{e}_1) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad T(\vec{e}_2) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

where $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Give a complete description of T.

Definition

Let $\vec{e}_j \in \mathbb{R}^n$ denote the vector having a 1 in the j^{th} row, and zeros elsewhere.

Theorem

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A so that for every $\vec{x} \in \mathbb{R}^n$,

$$T(\vec{x}) = A\vec{x}.$$

In fact, A is the $m \times n$ matrix whose j^{th} column is $T(\vec{e}_j)$.
-i.e.

$$A = \left(\begin{array}{cccc} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \ldots & T(\vec{e}_n) \\ | & | & & | \end{array} \right).$$

Example

Find the matrix representing the transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $T(\vec{x}) = 3\vec{x}$.

Definition

- A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be <u>onto</u> $\mathbb{R}^m$ if every $\vec{b} \in \mathbb{R}^m$ is the image of at least one $\vec{x} \in \mathbb{R}^n$. (T is onto provided that $\forall\ \vec{b} \in \mathbb{R}^m,\ \exists\ \vec{x} \in \mathbb{R}^n$ so that $T(\vec{x}) = \vec{b}$.)
- A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be <u>one-to-one</u> (or 1-1) if every $\vec{b} \in \mathbb{R}^m$ is the image of at most one $\vec{x} \in \mathbb{R}^n$. (That is, if $T(\vec{x}) = T(\vec{y})$ then $\vec{x} = \vec{y}$.)

Example

Suppose that T is the linear transformation with matrix

$$\begin{pmatrix} 1 & 2 & 5 & 6 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

1. If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, what are m and n?
2. Is T 1-1? What do you have to check?
3. Is T onto? What do you have to check?

Theorem

A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *is 1-1 if and only if* $T\vec{x} = \vec{0}$ *has only the trivial solution.*

Proof.

$\square$

Theorem

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation and let A be its matrix.

1. *T is onto if and only if the span of the columns of A is all of $\mathbb{R}^m$.*
2. *T is 1-1 if and only if the columns of A are linearly independent.*

Proof.

$\square$

NOTE

As a consequence of this theorem we can check whether T is 1-1 or onto by row-reducing its matrix A.
If there is a pivot in every row, then T is onto.
if there is a pivot in every column, then T is 1-1.

EXAMPLE

Let $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{pmatrix}$. Is T 1-1 and/or onto?