MTHSC 3110 Section 2.2 – Inverses of Matrices

Kevin James

Definition

Suppose that $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is linear. We will say that T is invertible if for every $\vec{b} \in \mathbb{R}^m$ there is exactly one $\vec{x} \in \mathbb{R}^n$ so that $T(\vec{x}) = \vec{b}$.

Note

If T is invertible, this means that T is onto (every equation can be solved: hence $m \leq n$) and T is 1-1 (every equation has at most one solution: hence $n \leq m$).
Thus $n = m$ and an invertible linear transformation has a matrix which must be square.

Questions

1. Which square matrices are invertible?
2. What does it mean for a square matrix to be invertible?

Fact

Suppose that $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *is an invertible linear transformation: then we can define* $S : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *so that* $T\vec{x} = \vec{u}$ *if and only if* $\vec{x} = S(\vec{u})$*. Furthermore, for every vector* $\vec{x} \in \mathbb{R}^n$, $S(T(\vec{x})) = \vec{x}$, *and for every* $\vec{u} \in \mathbb{R}^n$, $T(S(\vec{u})) = \vec{u}$.

Fact

It turns out that S *must also be linear.*

Proof.

We'll assume that $T(\vec{x}) = \vec{u}$ and $T(\vec{y}) = \vec{v}$. Then $S(\vec{u}) = \vec{x}$ and $S(\vec{v}) = \vec{y}$.

Note that $S(T(r\vec{x})) = r\vec{x}$, so we get

$$S(r\vec{u}) = S(rT(\vec{x})) = S(T(r\vec{x})) = r\vec{x} = rS(\vec{u})$$

so that S commutes with scalar addition.

Likewise,

$$S(\vec{u}+\vec{v}) = S(T(\vec{x})+T(\vec{y})) = S(T(\vec{x}+\vec{y})) = (\vec{x}+\vec{y}) = S(\vec{u})+S(\vec{v})$$

so that S commutes with addition.

Thus S is linear. □

Note

We see that if T is an invertible linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$, then so is S.
Hence we can represent T by a square matrix A and S by a square matrix B.
Then $S(T(\vec{x})) = \vec{x}$ for all $\vec{x}$ means that $BA\vec{x} = \vec{x}$ for every $\vec{x}$.
In particular, if $C = BA$, then we have $C\vec{e_j} = \vec{e_j}$, so that we obtain that C must be the identity matrix I_n.
Similarly, $T(S(\vec{u})) = \vec{u}$ for every $\vec{u}$, and hence $AB = I_n$ is also the identity matrix.

Definition

1. An $n \times n$ matrix A is *invertible* if there exists a $n \times n$ matrix B so that $AB = I_n = BA$.
2. B is called the *inverse* of A and is denoted by A^{-1}.
3. A matrix which is not invertible is said to be *singular*.

The Speical Case of 2×2 Matrices

Definition

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We define (and this only works for 2×2 matrices) the determinant of A to be the quantity

$$\det(A) = ad - bc.$$

Theorem

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *Then A is invertible if and only if* $\det(A)$ *is non-zero, in which case*

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

If $\det(A) = 0$ *then A is* singular.

Theorem

If A is an invertible $m \times m$ matrix, then for every $\vec{b} \in \mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has a unique solution, namely $\vec{x} = A^{-1}\vec{b}$.

Proof.

$\square$

Theorem

1. *If A is invertible, then so is A^{-1}, and $(A^{-1})^{-1} = A$.*
2. *If A and B are invertible $n \times n$ matrices then so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.*
3. *If A is invertible, then so is A^T, and $(A^T)^{-1} = (A^{-1})^T$.*

Proof.

$\square$

RECALL

Denoting rows r and s by R_r and R_s, the row operations are:

$R_r \leftrightarrow R_s$ Interchange rows R_r and R_s of a matrix.

cR_r For a non-zero $c \in \mathbb{R}$, replace R_r by cR_r.

$R_r + cR_s$ Replace R_r by $R_r + cR_s$

DEFINITION

An elementary matrix is any $n \times n$ matrix that can be obtained by performing a single elementary row operation to I_n.

Example

We construct three elementary matrices below.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2+2R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{3R_1} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example

Multiply the general 3×3 matrix on the left by each of the above matrices.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

Exercise

For a matrix having 4 rows, write down the elementary matrices which perform the following elementary row operations.

1. $R_1 \leftrightarrow R_3$
2. $3R_2$
3. $R_2 + 7R_4$

Exercise

Write down the inverse for each of the elementary matrices above.

Note

1. If $I_n \xrightarrow{\mathcal{R}} E$, then for any matrix A with n rows, $A \xrightarrow{\mathcal{R}} EA$.
2. So, if A can be row reduced to B by a sequence of row operations $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k$ and $I_n \xrightarrow{\mathcal{R}_i} E_i$, then
   $B = E_k E_{k-1} \cdots E_2 E_1 A$.
   (i.e. $A \xrightarrow{\mathcal{R}_1} E_1 A \xrightarrow{\mathcal{R}_2} E_2(E_1 A) \xrightarrow{\mathcal{R}_3} E_3 E_2 E_1 A \rightarrow \cdots \xrightarrow{\mathcal{R}_k} E_k E_{k-1} \cdots E_2 E_1 A = B$).
3. Since each row operation is invertible, each elementary matrix is invertible.

Theorem

An $n \times n$ matrix A is invertible if and only if $A \sim I_n$, in which case the sequence of elementary row operations which transform A to the identity also transform the identity matrix I_n to A^{-1}.

Note

Thus if $A \xrightarrow{\mathcal{R}_1} \xrightarrow{\mathcal{R}_2} \cdots \xrightarrow{\mathcal{R}_k} I_n$ then
$[A : I_n] \xrightarrow{\mathcal{R}_1} \xrightarrow{\mathcal{R}_2} \cdots \xrightarrow{\mathcal{R}_k} [I_n : A^{-1}]$.

Proof.

Recall that an $n \times n$ matrix A is invertible if and only if every equation $A\vec{x} = \vec{b}$ has a unique solution.
This is true if and only if the row reduced echelon form of A has a pivot in every row (existence of solution) and column (uniqueness of solution).
Thus A is invertible if and only if the row reduced echelon form of A is I_n.
Now suppose that A is invertible and that $A \xrightarrow{\mathcal{R}_1} \xrightarrow{\mathcal{R}_2} \cdots \xrightarrow{\mathcal{R}_k} I_n$.
Suppose also that $I_n \xrightarrow{\mathcal{R}_i} E_i$.
Then $A \xrightarrow{\mathcal{R}_1} E_1 A \xrightarrow{\mathcal{R}_2} E_2 E_1 A \rightarrow \cdots \xrightarrow{\mathcal{R}_k} E_k \cdots E_1 A = I_n$.
Thus $A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} \Rightarrow A^{-1} = E_k \cdots E_1$. □

Example

Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 3 & 5 & 1 \end{pmatrix}$. Find A^{-1}.