MTHSC 3110 Section 2.3 – Characterization of Invertible Matrices

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Theorem

Let A be an $n \times n$ matrix. The following are equivalent.

1. A *is invertible.*
2. $A \sim I_n$.
3. A *has* n *pivots.*
4. $A\underline{x} = \underline{0}$ *has only the trivial solution.*
5. *The columns of* A *are linearly independent.*
6. *The linear transformation* $T : \underline{x} \longmapsto A\underline{x}$ *is 1-1.*
7. *For every* $\underline{b} \in \mathbb{R}^n$, *the equation* $A\underline{x} = \underline{b}$ *has at least one solution.*
8. *The columns of* A *span* $\mathbb{R}^n$
9. *The linear transformation* $T : \underline{x} \longmapsto A\underline{x}$ *is onto.*
10. $\exists C$ *so that* $CA = I_n$.
11. $\exists D$ *so that* $AD = I_n$.
12. A^T *is invertible.*

Proof.

To prove a number of statements are equivalent, it is often easiest to show a chain of beginning and ending at one of the statements. Here, it is easiest to do the following chains:

$$(1) \implies (10) \implies (4) \implies (3) \implies (2) \implies (1)$$

$$(1) \implies (11) \implies (7) \implies (1)$$

$$(7) \iff (8) \iff (9)$$

$$(4) \iff (5) \iff (6)$$

$$(1) \iff (12)$$

$\square$

Example

Is the matrix $A = \begin{pmatrix} 0 & 7 & 1 \\ 1 & 2 & 4 \\ 2 & 3 & 6 \end{pmatrix}$ invertible?

Theorem

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation, and let A be the corresponding matrix. Then T is invertible if and only if A is invertible, in which case $T^{-1}\underline{x} = A^{-1}\underline{x}$.