

MTHSC 3110 SECTION 2.3 – CHARACTERIZATION OF INVERTIBLE MATRICES

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THEOREM

Let A be an $n \times n$ matrix. The following are equivalent.

- ① A is invertible.
- ② $A \sim I_n$.
- ③ A has n pivots.
- ④ $A\underline{x} = \underline{0}$ has only the trivial solution.
- ⑤ The columns of A are linearly independent.
- ⑥ The linear transformation $T : \underline{x} \mapsto A\underline{x}$ is 1-1.
- ⑦ For every $\underline{b} \in \mathbb{R}^n$, the equation $A\underline{x} = \underline{b}$ has at least one solution.
- ⑧ The columns of A span $\mathbb{R}^n$.
- ⑨ The linear transformation $T : \underline{x} \mapsto A\underline{x}$ is onto.
- ⑩ $\exists C$ so that $CA = I_n$.
- ⑪ $\exists D$ so that $AD = I_n$.
- ⑫ A^T is invertible.

PROOF.

To prove a number of statements are equivalent, it is often easiest to show a chain of beginning and ending at one of the statements. Here, it is easiest to do the following chains:

$$(1) \implies (10) \implies (4) \implies (3) \implies (2) \implies (1)$$

$$(1) \implies (11) \implies (7) \implies (1)$$

$$(7) \iff (8) \iff (9)$$

$$(4) \iff (5) \iff (6)$$

$$(1) \iff (12)$$



EXAMPLE

Is the matrix $A = \begin{pmatrix} 0 & 7 & 1 \\ 1 & 2 & 4 \\ 2 & 3 & 6 \end{pmatrix}$ invertible?

THEOREM

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation, and let A be the corresponding matrix. Then T is invertible if and only if A is invertible, in which case $T^{-1}\underline{x} = A^{-1}\underline{x}$.