MTHSC 3110 Section 3.1 – Determinants

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Exercise

Perform row reduction on the following matrices in order to characterize invertibility.

1. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

2. $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

Definition

Suppose that A is a 3×3 matrix as in the previous exercise. The determinant of A is defined by

$$\Delta(A) = aei + bfg + cdh - ceg - afh - dbi.$$

Fact

A is invertible if and only if $\Delta(A) \neq 0$.

Note

Suppose that A is a 3×3 matrix as before. Then,

$$\Delta(A) = a \cdot \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \cdot \det \begin{pmatrix} d & g \\ f & i \end{pmatrix} + c \cdot \det \begin{pmatrix} d & e \\ g & h \end{pmatrix}.$$

Definition

We define the ij^{th} minor of and $n \times n$ matrix A as

$$A_{ij} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1,n} \\ a_{2,1} & \cdots & a_{2,j-1} & a_{2,j+1} & \cdots & a_{2,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{n,n} \end{pmatrix}$$

Definition

Suppose that A is an $n \times n$ matrix. Then,

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \cdot \det(A_{1j}).$$

Example

Compute the det(A) where $A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 2 & 0 & 3 & 0 \\ 1 & 2 & 0 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}$.

Definition

We define the $(i,j)^{\text{th}}$ cofactor of A to be

$$C_{ij} = (-1)^{i+j}\det(A_{ij}).$$

Theorem

Suppose that A is an $n \times n$ matrix. Then we can compute the determinant of A by expanding by cofactors along any row or column of A. That is,

1. $\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}$ *i is fixed.*
2. $\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij}$ *j is fixed.*

Example

Compute $\det(A)$ where $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 1 \end{pmatrix}$.

Theorem

If A is triangular, then $\det(A) = a_{11} \cdot a_{22} \cdot \cdots \cdot a_{nn}$.