MTHSC 3110 Section 3.2 – Properties of Determinants

Kevin James

Theorem

Suppose that A and B are $n \times n$ matrices and that B is obtained from A by performing a single elementary row operation $\mathcal{R}$.

1. *If $\mathcal{R}$ is $R_i \leftrightarrow R_j$ then $\det(A) = -\det(B)$.*
2. *If $\mathcal{R}$ is $R_i + cR_j$ then $\det(A) = \det(B)$.*
3. *If $\mathcal{R}$ is cR_i then $\det(A) = \frac{1}{c}\det(B)$.*

Example

Compute $\det(A)$ where $A = \begin{pmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{pmatrix}$.

Corollary

1. $\det\left(E_{R_i \leftrightarrow R_j}\right) = -1.$
2. $\det\left(E_{R_i + cR_j}\right) = 1.$
3. $\det\left(E_{cR_j}\right) = c.$
4. *If* $A \xrightarrow{\mathcal{R}} B$, *then* $\det(B) = \det(E_{\mathcal{R}})\det(A).$

Note

Given an $n \times n$ matrix A, we can obtain a row echelon form of A, using only row replacement and row interchange operations.

Corollary

Suppose that A is an $n \times n$ matrix and that U is an echelon form of A obtained through the use of row replacements and row interchanges only. Let r denote the number of row interchanges used to obtain U. Then,

$$\begin{aligned} \det(A) &= (-1)^r \cdot u_{11} \cdot u_{22} \cdot \cdots \cdot u_{nn} \\ &= \begin{cases} (-1)^r * \textit{ product of pivots in } U & \textit{if } A \textit{ is invertible,} \\ 0 & \textit{if } A \textit{ in singular.} \end{cases} \end{aligned}$$

Corollary

Suppose that A is an $n \times n$ matrix. Then A is invertible if and only if $\det(A) \neq 0$.

Corollary

If A has two rows or two columns the same, then $\det(A) = 0$ and A is singular.

Proof.

$\square$

Note

When computing $\det(A)$, we can exploit the presence of many zeros in A.

Exercise

Compute $\det(A)$ where $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 8 & -1 & 5 \\ 1 & 2 & 3 & 4 \end{pmatrix}$.

Theorem

Suppose that A is an $n \times n$ matrix. Then A and its transpose have the same determinant (-i.e. $\det(A^T) = \det(A)$).

Theorem

Suppose that A and B are $n \times n$ matrices. Then

$$\det(AB) = \det(A)\det(B).$$

Example

Compute $\det(A)$, $\det(B)$ and $\det(AB)$ where $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$.