# MTHSC 3110 Section 3.3 – Cramer's Rule

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## Definition

Suppose that $A = [\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n}]$ is an $n \times n$ matrix. For any $\vec{b} \in \mathbb{R}^n$, we define

$$A_i(\vec{b}) = [\vec{a_1}, \ldots, \vec{a_{i-1}}, \vec{b}, \vec{a_{i+1}}, \ldots, \vec{a_n}].$$

## Theorem (Cramer's Rule)

*Suppose that $A$ is an $n \times n$ invertible matrix. For any $\vec{b} \in \mathbb{R}^n$, the unique solution to $A\vec{x} = \vec{b}$ has entries given by*

$$x_i = \frac{\det(A_i(\vec{b}))}{\det(A)}.$$

**Proof.**

We have

$$\begin{aligned} A \cdot I_i(\vec{x}) &= [A\vec{e_1}, A\vec{e_2}, \dots A\vec{e_{i-1}}, A\vec{x}, A\vec{e_{i+1}}, \dots, A\vec{e_n}] \\ &= [\vec{a_1}, \dots, \vec{a_{i-1}}, \vec{b}, \vec{a_{i+1}}, \dots, \vec{a_n}] \\ &= A_i(\vec{b}). \end{aligned}$$

So, we have $\det(A)\det(I_i(\vec{x})) = \det(A_i(\vec{b}))$.

Since $A$ is invertible, we may write $\det(I_i(\vec{x})) = \frac{\det(A_i(\vec{b}))}{\det(A)}$.

The theorem follows from noticing that $\det(I_i(\vec{x})) = x_i$.

To see this, compute $\det(I_i(\vec{x}))$ by expanding by cofactors along the $i^{\text{th}}$ row. □

### Example

Use Cramer's rule to solve $A\vec{x} = \vec{b}$ where $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 3 \\ 43 \end{pmatrix}$.

### Example

Consider the linear system

$$\begin{cases} 4sx_1 & + & 2x_2 & = & 1 \\ 5x_1 & + & x_2 & = & -1 \end{cases}$$

For which $s$ is there a unique solution. For such $s$ describe the solution.

## Definition

Suppose that $A$ is an $n \times n$ matrix. We define the $n \times n$ adjoint of $A$ as

$$\begin{aligned}
\operatorname{Adj}(A) &= \begin{pmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{pmatrix} \\
&= \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{pmatrix}^T,
\end{aligned}$$

where $C_{ij} = (-1)^{i+j} \det(A_{ij})$.

### Theorem

*Suppose that $A$ is an invertible $n \times n$ matrix. Then,*

$$A^{-1} = \frac{1}{\det(A)}\mathrm{Adj}(A).$$

### Proof.

$\square$

### Note

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\mathrm{Adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

## Exercise

Compute $\mathrm{Adj}(A)$ and $A^{-1}$ where $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 2 & 1 \end{pmatrix}$.

### Theorem

1. *If $A$ is a $2 \times 2$ matrix, then the area of the parallelogram determined by its columns (-i.e. having vertices at $\vec{0}$ at at the columns of $A$) is $|\det(A)|$.*
2. *If $A$ is a $3 \times 3$ matrix, then the volume of the parallelepiped determined by its columns is $|\det(A)|$.*

### Theorem

1. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with $2 \times 2$ matrix $A$. If $S$ is a parallelogram in $\mathbb{R}^2$, then $Area(T(S)) = |\det(A)| Area(S)$.*
2. *If $T : \mathbb{R}^3 \to \mathbb{R}^3$, is a linear transformation and $S$ is a parallelepiped, then $Vol(T(S)) = |\det(A)| Vol(S)$.*

### Note

The result of theorem 10, holds for any region $S$ of $\mathbb{R}^2$ for $\mathbb{R}^3$.

## Exercise

Suppose that $a, b \in \mathbb{N}$. Find the area bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 25.$$