MTHSC 3110 Section 3.3 – Cramer's Rule

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Definition

Suppose that $A = [\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n}]$ is an $n \times n$ matrix. For any $\vec{b} \in \mathbb{R}^n$, we define

$$A_i(\vec{b}) = [\vec{a_1}, \ldots, \vec{a_{i-1}}, \vec{b}, \vec{a_{i+1}}, \ldots, \vec{a_n}].$$

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Theorem (Cramer's Rule)

Suppose that A is an $n \times n$ invertible matrix. For any $\vec{b} \in \mathbb{R}^n$, the unique solution to $A\vec{x} = \vec{b}$ has entries given by

$$x_i = \frac{\det(A_i(\vec{b}))}{\det(A)}.$$

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So, we have $\det(A)\det(I_i(\vec{x})) = \det(A_i(\vec{b}))$.

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Since A is invertible, we may write $\det(I_i(\vec{x})) = \frac{\det(A_i(\vec{b}))}{\det(A)}$.

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To see this, compute $\det(I_i(\vec{x}))$ by expanding by cofactors along the i^{th} row. $\square$

Example

Use Cramer's rule to solve $A\vec{x} = \vec{b}$ where $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 3 \\ 43 \end{pmatrix}$.

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Example

Consider the linear system

$$\begin{cases} 4sx_1 & + & 2x_2 & = & 1 \\ 5x_1 & + & x_2 & = & -1 \end{cases}$$

For which s is there a unique solution. For such s describe the solution.

Definition

Suppose that A is an $n \times n$ matrix. We define the $n \times n$ adjoint of A as

$$\mathrm{Adj}(A) = \begin{pmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{pmatrix}$$

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where $C_{ij} = (-1)^{i+j} \det(A_{ij})$.

Theorem

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Note

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\mathrm{Adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Exercise

Compute $\mathrm{Adj}(A)$ and A^{-1} where $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 2 & 1 \end{pmatrix}$.

Theorem

1. *If A is a 2×2 matrix, then the area of the parallelogram determined by its columns (-i.e. having vertices at $\vec{0}$ at at the columns of A) is $|\det(A)|$.*
2. *If A is a 3×3 matrix, then the volume of the parallelepiped determined by its columns is $|\det(A)|$.*

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1. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then* $Area(T(S)) = |\det(A)| Area(S)$.
2. *If $T : \mathbb{R}^3 \to \mathbb{R}^3$, is a linear transformation and S is a parallelepiped, then* $Vol(T(S)) = |\det(A)| Vol(S)$.

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Note

The result of theorem 10, holds for any region S of $\mathbb{R}^2$ for $\mathbb{R}^3$.

Exercise

Suppose that $a, b \in \mathbb{N}$. Find the area bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 25.$$