MTHSC 3110 Section 4.1 – Vector Spaces and Subspaces

Kevin James

Goal

In this section, we generalize the notion of a vector space from the examples we've seen ($\mathbb{R}^n$), to include a number of other examples. As a result, we'll be able to apply tools from linear algebra (notions like linear independence, spanning sets, linear transformation, determinants) to these other examples.

Definition

A *vector space* V is a non-empty set, together with two operations, addition $+$ and scalar multiplication $\cdot$, satisfying

1. $\forall\ \vec{u}, \vec{v} \in V,\ \vec{u} + \vec{v} \in V$. (Closure under addition).
2. $\forall\ \vec{u}, \vec{v} \in V,\ \vec{u} + \vec{v} = \vec{v} + \vec{u}$ ($+$ is commutative).
3. $\forall\ \vec{u}, \vec{v}, \vec{w} \in V,\ \vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ ($+$ is associative).
4. $\exists \vec{0} \in V$ so that $\forall \vec{u} \in V,\ \vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$ (Additive identity).
5. $\forall \vec{u} \in V,\ \exists - \vec{u} \in V$ so that $\vec{u} + (-\vec{u}) = \vec{0}$ (Additive inverse).
6. For every $\vec{u} \in V$ and $c \in \mathbb{R}$, $c\vec{u} \in V$. (Closure under $\cdot$).
7. $\forall \vec{u}, \vec{v} \in V; c \in \mathbb{R},\ c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$. (Distributive Law).
8. $\forall \vec{u} \in V; c, d \in \mathbb{R},\ (c + d)\vec{u} = c\vec{u} + d\vec{u}$. (Distributive Law).
9. $\forall \vec{u} \in V; c, d \in \mathbb{R},\ c(d\vec{u}) = (cd)\vec{u}$ (Associativity of $\cdot$).
10. $\forall \vec{u} \in V,\ 1 \cdot \vec{u} = \vec{u}$. (Scalar Identity).

Note

There are a large number of conditions here. Checking whether a particular set V is a vector space requires checking all of them. As tedious as this may sometimes be, it is usually straightforward, and the major point is the following:

> *If the elements of a non-empty set V can be added together, multiplied by constants, and stay in V, and things work nicely, then V is a vector space.*

Fact

For every $\vec{u} \in V$ and $c \in \mathbb{R}$

1. $0\vec{u} = \vec{0}$
2. $c\vec{0} = \vec{0}$
3. $-\vec{u} = (-1)\vec{u}$

Note

$-\vec{u}$ refers to the additive inverse of the vector $\vec{u}$. This shows that we *can* choose to interpret it as (-1) times the vector $\vec{u}$.

Example

Let $n \geq 0$ be an integer. Let

$$\mathbb{P}_n = \{a_0 + a_1 t + \cdots + a_n t^n \mid a_i \in \mathbb{R}\}$$

be the set of polynomials of degree at most n.
The *degree* of $p(t)$ is the highest power of t whose coefficient is not zero.
If $p(t) = a_0 \neq 0$, then the degree of $p(t)$ is zero.
If all the coefficients of $p(t)$ are zero, then we call $p(t)$ the *zero polynomial*. Its degree is technically speaking undefined, but we include it in the set $\mathbb{P}_n$ too.
We can add two polynomials.
We can multiply a polynomial by a scalar.
The set $\mathbb{P}_n$ is a vector space. The zero polynomial is the zero vector.

Example

Let $\mathbb{P}$ be the set of all polynomials, that is $\mathbb{P} = \bigcup_{n \geq 0} \mathbb{P}_n$. Then $\mathbb{P}$ is also a vector space. Note also that $\mathbb{P}_0 \subseteq \mathbb{P}_1 \subseteq \mathbb{P}_2 \dots$ and for each $n \geq 0$, $\mathbb{P}_n \subseteq \mathbb{P}$.

Example

Let

$$C((0,1)) = \{f : (0,1) \to \mathbb{R} \mid f \text{ is continuous}\}.$$

We can define addition and scalar multiplication on $C((0,1))$ as follows.

$$(f+g)(x) = f(x) + g(x); \qquad (cf)(x) = cf(x).$$

We can check that $C(0,1)$ is also a vector space. Here the zero vector is the function which is zero on the interval $(0,1)$.

Example

Let

$$V = \left\{ f \in C((0,1)) \ : \ \int_0^1 f(t)\, dt = 0 \right\}.$$

Then the sum of two functions with integral zero is a function whose integral is zero.
If we multiply f by a scalar, we still get a function whose integral is zero.
Addition and multiplication "work nicely", so this is probably a vector space.
Check that V is a vector space.
What is the zero vector?

Example

Let $V = \{f \in C((0,1)) \mid f(1/2) = 0\}$. Is this a vector space?

Example

Let $V = \{f \in C((0,1))\ f(1/2) = 1\}$. Is this a vector space?

Example

Let V be the set of polynomials of degree exactly n. Is this a vector space?

Example

Let $\mathbb{M}_{m,n}$ denote the set of $m \times n$ matrices. Does this form a vector space?

DEFINITION

If V is a vector space with respect to $+$ and $\cdot$, with zero vector $\vec{0}$, then a set $H \subseteq V$ is a *subspace* of V if

1. $\vec{0} \in H$
2. For every $\vec{u}, \vec{v} \in H$, $\vec{u} + \vec{v} \in H$.
3. For every $\vec{u} \in H$ and $c \in \mathbb{R}$, $c\vec{u} \in H$.

EXAMPLE

For any vector space V with zero vector $\vec{0}$, the set $\{\vec{0}\}$ is a subspace of V.

Example

If $m \leq n$ the $\mathbb{P}_m$ is a subspace of $\mathbb{P}_n$.

Example

Let $V = C((0,1))$ and let $H = \{f \in C((0,1)) \mid f(1/2) = 0\}$. Then H is a subspace of V.

Note

$\mathbb{R}^2$ is *not* a subspace of $\mathbb{R}^3$. Indeed, $\mathbb{R}^2$ is not even a *subset* of $\mathbb{R}^3$. However, a plane through the origin in $\mathbb{R}^3$ *is* a subspace of $\mathbb{R}^3$.

Definition

Suppose that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in V$ and $c_1, c_2, \ldots c_k \in \mathbb{R}$. Then

$$\sum_{i=1}^{k} c_i \vec{v}_i$$

is the linear combination of $\vec{v}_1, \ldots, \vec{v}_k$ with weights $c_1, \ldots c_k$.

Definition

$\mathrm{Span}(\vec{v}_1, \ldots, \vec{v}_k)$ denotes the set of all linear combinations of $\vec{v}_1, \ldots, \vec{v}_k$.

Theorem

If V is a vector space and if $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in V$, then $H = Span(\vec{v}_1, \ldots, \vec{v}_k)$ is a subspace of V.