MTHSC 3110 Section 4.2 – Null Spaces, Column Spaces and Linear Transformations

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Definition

Let A be an $m \times n$ matrix. We define the *null space* of A as follows.

$$\text{Nul } A = \{\vec{x} \in \mathbb{R}^n \mid \text{ and } A\vec{x} = \vec{0}\}.$$

Theorem

If A is an $m \times n$ matrix, then Nul A is a subspace of $\mathbb{R}^n$.

Proof.

$\square$

Example

Let $H = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \middle| \; a - 2b + 5c = d \text{ and } c - a = b \right\}$. Show that H is a subspace of $\mathbb{R}^4$ by expressing this as a null space of a matrix. Find a spanning set for this H.

Example

Find a spanning set for the null space of

$$A = \begin{pmatrix} 1 & -2 & 2 & -3 & -1 \\ 2 & -4 & 5 & -6 & -3 \\ -3 & 6 & -4 & 1 & -7 \end{pmatrix}$$

Finding the Spanning Set for Nul(A)

1. Solve $A\vec{x} = \vec{0}$ and express the answer in vector parametric form.
2. Recall that the non pivot columns correspond to free variables, say $x_{i_1}, x_{i_2}, \ldots x_{i_k}$.
3. The solution set can then be expressed as

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_{i_1} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + x_{i_2} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \cdots + x_{i_k} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

4. Note that the vectors on the right span Nul(A).

Note

If $\mathrm{Nul}(A) \neq \{\vec{0}\}$, then the vectors in our construction of a spanning set form a linearly independent set.

Proof.

Let's call the vector on the right appearing next to x_{i_j} $\vec{v_{i_j}}$.
Note that the i_j entry in the vectors on the right is 0 except in $\vec{v_{i_j}}$.
This vector has a 1 in the i_j position.
So, if we have a dependence, say $\vec{0} = \sum_{m=1}^{k} w_{i_m} \vec{v_{i_m}}$,
we can consider only the i_j^{th} entries to obtain
$0 = \sum_{m=1}^{k} w_{i_m} [\vec{v_{i_m}}]_{i_j} = w_{i_j}$.
Since this is true for $1 \leq j \leq k$, we see that the dependence must be the trivial one.
So, $\{\vec{v_{i_1}}, \ldots \vec{v_{i_k}}\}$ is an independent set. $\square$

Summary

Our construction of a spanning set for Nul(A) produces a set of vectors which spans Nul(A) and is linearly independent. Further, if Nul(A) $\neq \{\vec{0}\}$, then the size of our spanning set is the number of free variables which is in turn equal to the number of non pivot columns.

Definition

Let A be an $m \times n$ matrix having column form $[\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n]$. Then the column space of A, denoted Col A is given by

$$\text{Col } A = \text{Span}(\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n).$$

Theorem

If A is an $m \times n$ matrix, then Col A is a subspace of $\mathbb{R}^m$.

Proof.

Note that Col $A = \{A\vec{x} \; : \; \vec{x} \in \mathbb{R}^n\}$, since any linear combination of the columns of A with weights $x_1, x_2, \ldots, x_n$ is of this form.
Clearly Col $A \subseteq \mathbb{R}^m$ (since the columns of A are in this space, so are all linear combinations of them).
To show that Col A is a subspace of $\mathbb{R}^m$, we have to show

1. $\vec{0} \in$ Col A.
2. If $\vec{u}, \vec{v} \in$ Col A then $\vec{u} + \vec{v} \in$ Col A.
3. If $c \in \mathbb{R}$ and $\vec{u} \in$ Col A then $c\vec{u} \in$ Col A.

$\square$

Example

Find a matrix A so that Col $A = \left\{ \begin{pmatrix} 5a - b \\ 3b + 2a \\ -7a \end{pmatrix} \ : \ a, b \in \mathbb{R} \right\}$

Note

If $A = [\vec{a_1}, \ldots, \vec{a_n}]$, then Col$(A)$ is spanned by $\{\vec{a_1}, \ldots, \vec{a_n}\}$. How do we find a linearly independent spanning set for Col(A)?

Note

For an $m \times n$ matrix A, Col $A = \mathbb{R}^m$

$\iff$ if for every $\vec{b} \in \mathbb{R}^m$ the equation $A\vec{x} = \vec{b}$ has a solution

$\iff$ if the linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ with matrix A is onto.

Notation

If W is a subspace of a vector space V, then we will write $W \leq V$ or $W < V$ if $W \neq V$.

Example

Let $A = \begin{pmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{pmatrix}$

1. Col $A \leq$ ________________
2. Nul $A \leq$ ________________
3. Give a vector in Col A.

Example continued ...

Note that A row reduces to $\begin{pmatrix} 1 & 0 & 9 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

1. Describe Nul A in vector parametric form.
2. Let $\vec{u} = \begin{pmatrix} 3 \\ -2 \\ -1 \\ 0 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}$. Is either of $\vec{u}$ or $\vec{v}$ in either Col A or Nul A?

Definition

Suppose that U and V are vector spaces. A transformation $T : U \longrightarrow V$ is said to be linear if

1. For all $\vec{u}, \vec{v} \in U$, $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
2. For all $\vec{u} \in U$ and $c \in \mathbb{R}$, $T(c\vec{u}) = cT(\vec{u})$.

Definition

Let U and V be vector spaces, and let $T : U \longrightarrow V$ be a linear transformation. The *kernel* of T is

$$\ker(T) := \{\vec{u} \in U \ : \ T(\vec{u}) = \vec{0}\}.$$

The *range* of T is

$$\text{range}(T) = \text{im}(T) := \{T(\vec{u}) \ : \ \vec{u} \in U\}.$$

Fact

Given a linear transformation $T : U \longrightarrow V$,

1. $ker(T) \leq U$.
2. $im(T) \leq V$.

Proof.

$\square$