MTHSC 3110 Section 4.3 – Linear Independence in Vector Spaces; Bases

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Definition

1. Let V be a vector space and let $\{\vec{v}_1.\vec{v}_2,\ldots,\vec{v}_p\} \subseteq V$. If the only solution to the equation
$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = \vec{0}$$
is the trivial solution $x_1 = x_2 = \cdots = x_p = 0$ then $\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_p\}$ is said to be *linearly independent*.
2. If there *is* a non-trivial solution to the equation then the set of vectors is said to be *linearly dependent*.

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2. If there *is* a non-trivial solution to the equation then the set of vectors is said to be *linearly dependent*.

Theorem

Let V be a vector space. Suppose that $\{\vec{v}_1,\vec{v}_2,\ldots\vec{v}_p\} \subseteq V$, $p \geq 2$, and that $\vec{v}_1 \neq \vec{0}$. Then $\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_p\}$ is linearly dependent if and only if there is a $1 \leq j \leq p$ so that $\vec{v}_j$ is a linear combination of the vectors $\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_{j-1}$.

Example

Consider the vector space $\mathbb{P}_3$. Let

$$S = \{x^2 + 2x + 3,\ x^3 + 1,\ x^3 + 2x^2 + 4x + 7\}.$$

$S \subseteq \mathbb{P}_3$. Is it linearly dependent or linearly independent?

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Example

Consider the vector space

$$V = \{f : [0,1] \longrightarrow \mathbb{R} \text{ so that } f \text{ is continuous}\}$$

and let $S = \{\sin(x), \cos(x)\} \subseteq V$. Is this set linearly dependent or linearly independent?

Definition

Suppose that V is a vector space with $W \leq V$, and that $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\} \subseteq W$. This ordered set of vectors is a *basis* for W provided that

1. $\mathcal{B}$ is linearly independent, and
2. $\text{Span}(\mathcal{B}) = W$.

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Example

Let $A = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n]$ be an invertible $n \times n$ matrix. What does the Invertible Matrix Theorem say about $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$?

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Example

$\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is a basis for $\mathbb{R}^n$. Why?

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$\{1, x, x^2, \ldots, x^n\}$ is a basis for $\mathbb{P}_n$. Why?

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Example

Let

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

Show that $\text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \text{Span}(\vec{v}_1, \vec{v}_2)$.

Theorem

Let V be a vector space, and $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\} \subseteq V$, and let $H = Span(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p)$.

1. *If there exists k so that $\vec{v}_k$ is a linear combination of the other vectors in S, then*

$$H = Span(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{k-1}, \vec{v}_{k+1}, \ldots, \vec{v}_p)$$

2. *If $H \neq \{\vec{0}\}$ then some subset of S is a basis for H.*

Proof.

1. Suppose that $\vec{u} \in H$. We need to show that $\vec{u}$ is a linear combination of vectors in $\vec{v}_1, \ldots, \vec{v}_{k-1}, \vec{v}_{k+1} \ldots, \vec{v}_n$. We know that it is a linear combination of vectors in S.

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2. If S is linearly independent, then it is a basis. Otherwise, there is a non-trivial linear combination of vectors in S giving $\vec{0}$, and hence there is some vector in S which can be written as a linear combination of the others. Hence we can replace S by a smaller set S' which still spans H. Clearly we can continue this process, and it has to stop either with $S' = \emptyset$ (in which case $H = \{\vec{0}\}$) or with S' a linearly independent set spanning H, and hence a basis for H.

$\square$

Note

We already have seen how to find a basis for Nul A: row reduce A to obtain a matrix in reduced row echelon form and use this to express the null space in vector parametric form. The vectors appearing will be the basis for Nul A.

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Example

Let

$$B = \begin{pmatrix} 1 & 3 & 0 & 2 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Find a basis for Nul B.

Example

Again, let

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Fact

If $A \sim B$, then the linear dependencies of the columns of A are exactly the same as the linear dependencies of the columns of B.

As a result, we have

Theorem

The pivot columns of a matrix A form a basis for Col A.

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A basis is

- A spanning set which is as small as possible
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Example

Which of the following sets of vectors form a basis for $\mathbb{R}^3$.

1. $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right\}$
2. $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$
3. $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$