MTHSC 3110 Section 4.4 – Coordinate Systems

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Theorem (Unique Representation Theorem)

Let V be a vector space, and let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ be a basis for V. Then for every $\vec{v} \in V$, there is a unique vector $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ *so that*

$$\vec{v} = x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_n\vec{b}_n.$$

Proof.

Since $\mathcal{B}$ is a basis, we know that there is at least one representation of the form.

$$\vec{v} = x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_n\vec{b}_n.$$

Now, suppose that we also have

$$\vec{v} = y_1\vec{b}_1 + y_2\vec{b}_2 + \cdots + y_n\vec{b}_n.$$

Subtracting the two expressions for $\vec{v}$, we obtain

$$\vec{0} = (x_1 - y_1)\vec{b}_1 + (x_2 - y_2)\vec{b}_2 + \cdots + (x_n - y_n)\vec{b}_n.$$

Since $\mathcal{B}$ is linearly independent, the weights $(x_1 - y_1), (x_2 - y_2), \ldots, (x_n - y_n)$ must be 0.
Hence $x_i = y_i$ for $1 \leq i \leq n$ and the two representations are in fact the same. $\square$

Notation

Suppose $\mathcal{B}$ and V are as above. Given a vector $v \in V$ we can write

$$v = \sum_{i=1}^{n} x_i \vec{b}_i = x_1 \vec{b}_1 + x_2 \vec{b}_2 + \cdots + x_n \vec{b}_n.$$

We define the coordinate vector of v with respect to the basis $\mathcal{B}$ as

$$[v]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Example

Suppose that $\vec{b}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{b}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ is a basis for $\mathbb{R}^2$. Suppose that $\vec{x}$ has a coordinate representation with respect to this basis

$$[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} 5 \\ -7 \end{pmatrix}.$$

Then $x =$

Suppose $\vec{x} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$.

Then $[\vec{x}]_{\mathcal{B}} =$

Suppose that $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_p\}$ is a basis for $\mathbb{R}^n$.
Let $P_{\mathcal{B}} = [\vec{b}_1, \ldots, \vec{b}_p]$.
Given $\vec{v} \in \mathbb{R}^n$, we can write (uniquely)

$$\vec{v} = w_1 \vec{b}_1 + \cdots + w_p \vec{b}_p = P_{\mathcal{B}} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_p \end{pmatrix}.$$

That is $P_{\mathcal{B}}\vec{x} = \vec{v}$ has a unique solution for each $\vec{v} \in \mathbb{R}^n$.
Thus when we row reduce $P_{\mathcal{B}}$ we must have a pivot in each row (existence of solutions) and a pivot in each column (uniqueness).
Thus $P_{\mathcal{B}}$ must be $n \times n$ (-i.e. $p = n$) and invertible.

Suppose that $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ is a basis for $\mathbb{R}^n$ and let $P_{\mathcal{B}} = [\vec{b}_1, \ldots, \vec{b}_n]$.
Then the vector equation

$$\vec{v} = x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_p\vec{b}_n.$$

corresponds to the matrix equation

$$\vec{v} = P_{\mathcal{B}}\vec{x}.$$

But the weights $x_1, x_2, \ldots, x_n$ in $\vec{x}$ are precisely the coordinates of $\vec{v}$ with respect to the basis $\mathcal{B}$. Hence

$$\vec{v} = P_{\mathcal{B}}[\vec{v}]_{\mathcal{B}} \qquad \text{and} \qquad [\vec{v}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\vec{v}.$$

Definition

$P_{\mathcal{B}}$ is called the *change of coordinates* matrix.

Theorem

Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a basis for a vector space V. Let $T : V \to \mathbb{R}^n$ be defined by

$$T(v) = [v]_{\mathcal{B}}.$$

Then T is a one-to-one *linear transformation* onto $\mathbb{R}^n$.

Note

We have assumed that the number of vectors in the basis is equal to n, the dimension of $\mathbb{R}^n$.

Proof.

It is a linear transformation:

It is one-to-one

It is onto

$\square$

Definition

If V has basis $\mathcal{B} = \{b_1, \ldots, b_n\}$ as above, then we say that V is isomorphic to $\mathbb{R}^n$ (isomorphic meaning "same shape" or "same form"), which we write as $V \cong \mathbb{R}^n$.

Corollary

For any vectors $v, v_1, \ldots, v_k \in V$,

$$[v]_{\mathcal{B}} = \vec{0} \qquad \iff \qquad v = 0$$

and

$$[c_1 v_1 + c_2 v_2 + \cdots + c_k v_k]_{\mathcal{B}} = c_1 [v_1]_{\mathcal{B}} + c_2 [v_2]_{\mathcal{B}} + \cdots + c_k [v_k]_{\mathcal{B}}.$$

Example

$\mathbb{P}_3$ has basis $\mathcal{B} = \{1, x, x^2, x^3\}$. If

$$p(x) = ax^3 + bx^2 + cx + d \in \mathbb{P}_3$$

then

$$[p(x)]_{\mathcal{B}} = \left(\qquad \right).$$

$\mathbb{P}_3 \cong$ ________.

Example

$\mathcal{B}' = \{1, x, x^2 - x, x^3 - 3x^2 + 2x\}$ is also a basis for $\mathbb{P}_3$. For the same $p(x)$ above, compute $[p(x)]_{\mathcal{B}'}$.

Example

Consider the set $S \subset \mathbb{P}_3$.

$$S = \left\{ \begin{matrix} p(x) = 1 + x + x^3, q(x) = 2 + x^2, \\ r(x) = 4 + 2x + x^2 + 2x^3, s(x) = 1 + x + x^2 + x^3 \end{matrix} \right\}.$$

Is S linearly dependent or linearly independent?

Example

Let $\vec{v}_1 = \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \vec{x} = \begin{pmatrix} 3 \\ 12 \\ 7 \end{pmatrix}$ Then $\{\vec{v}_1, \vec{v}_2\}$ is a basis for $H = \text{span}(\vec{v}_1, \vec{v}_2)$. Is $\vec{x} \in H$ and if so, what is $[\vec{x}]_{\mathcal{B}}$?