MTHSC 3110 Section 4.5 – The Dimension of a Vector Space

Kevin James

Theorem

Suppose that V is a vector space with basis $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ and suppose that $S \subseteq V$. If $\#S > n$ then S is linearly dependent.

Proof.

Suppose that $p > n$ and $\{u_1, u_2, \ldots, u_p\} \subseteq V$.
Then $\{[u_1]_{\mathcal{B}}, [u_2]_{\mathcal{B}}, \ldots, [u_p]_{\mathcal{B}}\} \subseteq \mathbb{R}^n$ is LD since $p > n$.
So there are $c_1, c_2, \ldots, c_p \in \mathbb{R}$, not all zero, so that

$$c_1[u_1]_{\mathcal{B}} + c_2[u_2]_{\mathcal{B}} + \cdots + c_p[u_p]_{\mathcal{B}} = \vec{0}.$$

Since the coordinate mapping is a linear transformation,

$$[c_1u_1 + c_2u_2 + \cdots + c_pu_p]_{\mathcal{B}} = \vec{0}$$

Since $[\cdot]_{\mathcal{B}}$ is 1-1, $c_1u_1 + c_2u_2 + \cdots + c_pu_p = 0_V$, and since not all of the c_i are zero, the vectors are linearly dependent. □

Theorem

If V is a vector space with a basis of size n, then every basis for V has exactly n vectors.

Proof.

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bases having n and p vectors respectively.
First, since $\mathcal{B}_1$ is a basis, and $\mathcal{B}_2$ is linearly independent, from the previous theorem we know that $p \leq n$.
Similarly, since $\mathcal{B}_2$ is a basis, and $\mathcal{B}_1$ is linearly dependent, $n \leq p$.
Thus $p \leq n \leq p$ and we see that $p = n$. $\square$

Recall

If V is spanned by a finite set, then by repeatedly discarding vectors which are part of a non-trivial dependence, we can find a finite basis for V. This theorem says that every basis must have the same number of vectors in it.

Definition

If V is spanned by a finite set, then V is said to be <u>finite dimensional</u>, and the <u>dimension</u> of V, written as $\dim(V)$, is the number of vectors in a basis for V.
The dimension of the zero vector space $\{0\}$ is defined to be 0.
If V is not spanned by a finite set, then V is said to be <u>infinite dimensional</u>.

Example

$$\dim \mathbb{R}^n = \underline{\qquad\quad}$$
$$\dim \mathbb{P}_n = \underline{\qquad\quad}$$
$$\dim \mathbb{P} = \underline{\qquad\quad}$$

Example

Find the dimension of the subspace

$$H = \left\{ \begin{pmatrix} a + 4b + c + 2d \\ a + 2b + d \\ a + 5b + c + 3d \\ b + d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

Theorem

If V is a finite-dimensional vector space, and if $H \leq V$, then any linearly independent set $S \subset H$ can be expanded to a basis for H and

$$\dim(H) \leq \dim(V).$$

Proof.

Key Idea: If $S = \{u_1, \ldots, u_k\}$ and if $H \neq \mathrm{Span}(S)$, then there is a vector $u_{k+1} \in H - \mathrm{Span}(S)$, which implies $\{u_1, \ldots, u_k, u_{k+1}\}$ is LI.
We can continue enlarging S as long as it doesn't span H.
Note that in at most $\dim(V) - 1$ steps, we have $\#S = \dim(V)$ which is the maximal size of a LI subset of V.
Thus at this point we would have $H \leq V = \mathrm{Span}(S)$.
So, our enlarging process must stop in a finite number of steps, producing a basis for H.
Since S is then a basis for H and hence a LI subset of V, $\#S \leq \dim(V)$.
So, $\dim(H) = \#S \leq \dim(V)$. $\square$

Theorem

Suppose that V is a p-dimensional vector space. Then

1. *Any linearly independent set of p vectors in V is a basis for V.*
2. *Any set of p vectors which spans V is a basis for V.*

Proof.

First, suppose that S is a linearly independent set of size p.
For any $v \in V$, if v is included in S then, S would become LD.
Thus $v \in \text{Span}(S)$.
So, $V = \text{Span}(S)$.
Now, suppose that $V = \text{Span}(S)$ and that S has size p.
Then S contains a basis for V.
This basis must have size $p = \#S$ and thus must be all of S. □

Theorem

$$\dim(\mathit{Col}(A)) = \textit{ number of pivots in } A$$

$$\dim(\mathit{Nul}(A)) = \textit{ number of free variables in rref of } A$$

Note

The dimension of the null space is the number of columns of A minus the number of pivots.
Hence we have

$$\dim(\mathrm{Col}(A)) + \dim(\mathrm{Nul}(A)) = \text{ number of columns of } A.$$

Example

Let

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

What is $\dim(\mathrm{Nul}(A))$ and $\dim(\mathrm{Col}(A))$?

$\dim(V) = 0$: $V =$ ____________________________.

$\dim(V) = 1$: $V =$ ____________________________.

$\dim(V) = 2$: $V =$ ____________________________.

$\dim(V) = 3$: $V =$ ____________________________.