MTHSC 3110 Section 4.5 – The Dimension of a Vector Space

Kevin James

Theorem

Suppose that V is a vector space with basis $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ and suppose that $S \subseteq V$. If $\#S > n$ then S is linearly dependent.

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So there are $c_1, c_2, \ldots, c_p \in \mathbb{R}$, not all zero, so that

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Since $[\cdot]_{\mathcal{B}}$ is 1-1, $c_1u_1 + c_2u_2 + \cdots + c_pu_p = 0_V$, and since not all of the c_i are zero, the vectors are linearly dependent. $\square$

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Thus $p \leq n \leq p$ and we see that $p = n$. $\square$

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If V is not spanned by a finite set, then V is said to be <u>infinite dimensional</u>.

Example

$$\dim \mathbb{R}^n = \underline{\qquad\qquad}$$

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Example

Find the dimension of the subspace

$$H = \left\{ \begin{pmatrix} a + 4b + c + 2d \\ a + 2b + d \\ a + 5b + c + 3d \\ b + d \end{pmatrix} \ : \ a, b, c, d \in \mathbb{R} \right\}$$

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Key Idea: If $S = \{u_1, \ldots, u_k\}$ and if $H \neq \text{Span}(S)$, then there is a vector $u_{k+1} \in H - \text{Span}(S)$, which implies $\{u_1, \ldots, u_k, u_{k+1}\}$ is LI.

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$\#S \le \dim(V)$.
So, $\dim(H) = \#S \le \dim(V)$. $\square$

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Now, suppose that $V = \text{Span}(S)$ and that S has size p.
Then S contains a basis for V.
This basis must have size $p = \#S$ and thus must be all of S. $\square$

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$$\dim(\mathit{Col}(A)) = \textit{ number of pivots in } A$$

$$\dim(\mathit{Nul}(A)) = \textit{ number of free variables in rref of } A$$

The dimensions of Nul(A) and Col(A)

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Hence we have

$$\dim(\mathrm{Col}(A)) + \dim(\mathrm{Nul}(A)) = \text{ number of columns of } A.$$

Example

Let

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

What is $\dim(\mathrm{Nul}(A))$ and $\dim(\mathrm{Col}(A))$?

$\dim(V) = 0$: $V =$ ______________________________.

$\dim(V) = 1$: $V =$ ______________________________.

$\dim(V) = 2$: $V =$ ______________________________.

$\dim(V) = 3$: $V =$ ______________________________.