MTHSC 3110 Section 4.5 – The Dimension of a Vector Space

Kevin James

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Suppose that V is a vector space with basis $\mathcal{B} = \{b_1, b_2, ..., b_n\}$ and suppose that $S \subseteq V$. If #S > n then S is linearly dependent.

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$$c_1[u_1]_{\mathcal{B}}+c_2[u_2]_{\mathcal{B}}+\cdots+c_p[u_p]_{\mathcal{B}}=\vec{0}.$$

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Since $[\cdot]_{\mathcal{B}}$ is 1-1, $c_1u_1 + c_2u_2 + \cdots + c_pu_p = 0_V$, and since not all of the c_i are zero, the vectors are linearly dependent.

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If V is spanned by a finite set, then by repeatedly discarding vectors which are part of a non-trivial dependence, we can find a finite basis for V. This theorem says that every basis must have the same number of vectors in it.

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If V is spanned by a finite set, then V is said to be <u>finite dimensional</u>, and the <u>dimension</u> of V, written as $\dim(V)$, is the number of vectors in a basis for V.

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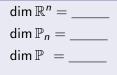
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If V is spanned by a finite set, then V is said to be finite dimensional, and the dimension of V, written as dim(V), is the number of vectors in a basis for V. The dimension of the zero vector space $\{0\}$ is defined to be 0. If V is not spanned by a finite set, then V is said to be infinite dimensional.

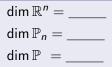
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EXAMPLE



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EXAMPLE



EXAMPLE

Find the dimension of the subspace

$$H = \left\{ \begin{pmatrix} a+4b+c+2d\\a+2b+d\\a+5b+c+3d\\b+d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

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If V is a finite-dimensional vector space, and if $H \le V$, then any linearly independent set $S \subset H$ can be expanded to a basis for H and

 $\dim(H) \leq \dim(V).$

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Key Idea: If $S = \{u_1, \ldots, u_k\}$ and if $H \neq \text{Span}(S)$, then there is a vector $u_{k+1} \in H - \text{Span}(S)$, which implies $\{u_1, \ldots, u_k, u_{k+1}\}$ is LI.

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Suppose that V is a p-dimensional vector space. Then

- **1** Any linearly independent set of p vectors in V is a basis for V.
- 2 Any set of p vectors which spans V is a basis for V.

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The dimensions of Nul(A) and Col(A)

Theorem

dim(Col(A)) = number of pivots in Adim(Nul(A)) = number of free variables in rref of A

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Note

The dimension of the null space is the number of columns of A minus the number of pivots.

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Hence we have

 $\dim(Col(A)) + \dim(Nul(A)) =$ number of columns of A.

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EXAMPLE

Let

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$$
$$\rightarrow \begin{pmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\rightarrow \begin{pmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$
What is dim(Nul(A)) and dim(Col(A))?

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SUBSPACES OF \mathbb{R}^n

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$$dim(V) = 0: V = ______.$$

$$dim(V) = 1: V = _____.$$

$$dim(V) = 2: V = _____.$$

$$dim(V) = 3: V = ____.$$