MTHSC 3110 Section 4.6 – The Rank of a Matrix

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Note

Suppose that A is an $m \times n$ matrix. We can view A as a collection of *rows* instead of a collection of columns.

Note

The set of length n row vectors with real entries is a vector space.

Definition

We define $\text{Row}(A)$ to be the span of the rows of A.

Theorem

If $A \sim B$, then Row(A)=Row(B). If B is in echelon form, then the non-zero rows of B are a basis for Row(B)= Row(A).

Sketch of Proof.

Suppose that $A \sim B$.
Then the rows of B are linear combinations of the rows of A and vice versa.
Thus $\mathrm{Row}(A) = \mathrm{Row}(B)$.
It is fairly easy to see that the nonzero rows of a matrix in echelon form are linearly independent and span the row space of that matrix. $\square$

Corollary

$\dim(Row(A)) = \#\ of\ pivots = \dim(Col(A))$.

Example

Let

$$A = \begin{pmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Find Row(A), Col(A), and Nul(A).

Theorem (The Rank-Nullity Theorem)

Let A be an $m \times n$ matrix. Then

$$\dim(Row(A)) = \dim(Col(A)).$$

We call this value rank(A) and further, we have

$$rank(A) + \dim(Nul(A)) = n.$$

Proof.

We have seen that $\dim(\text{Row}(A)) = \dim(\text{Col}(A)) = \#\text{pivots}$.
We will call this common value $\text{rank}(A)$.
Recall that $\dim(\text{Nul}(A)) = \#$ free variables $= n - \#$ pivots..
Thus

$$\dim(\text{Nul}(A)) + \text{rank}(A) = (n - \#\text{ pivots}) + \#\text{pivots} = n.$$

$\square$

Theorem

Let A be an $n \times n$ matrix. The following (extra) conditions are equivalent to A being invertible:

(M) *The columns of A are a basis for $\mathbb{R}^n$.*

(N) $Col(A) = \mathbb{R}^n$.

(O) $\dim(Col(A)) = n$.

(P) $rank(A) = n$.

(Q) $Nul(A) = \{\vec{0}\}$.

(R) $\dim(Nul(A)) = 0$.