MTHSC 3110 Section 4.7 – Change of Basis

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Example

Suppose that we have two bases $\mathcal{B} = \{\underline{b}_1, \underline{b}_2\}$ and $\mathcal{C} = \{\underline{c}_1, \underline{c}_2\}$ for a 2-dimensional vector space V. Since $\mathcal{C}$ is a basis, we can express $\mathcal{B}$ in terms of $\mathcal{C}$. Suppose that we know

$$\underline{b}_1 = 4\underline{c}_1 + 3\underline{c}_2 \qquad \text{and} \qquad \underline{b}_2 = 2\underline{c}_1 + \underline{c}_2.$$

Suppose $[\underline{x}]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

Find $[\underline{x}]_{\mathcal{C}}$.

Theorem

Let $\mathcal{B} = \{\underline{b}_1, \ldots, \underline{b}_n\}$ and $\mathcal{C} = \{\underline{c}_1, \ldots, \underline{c}_n\}$ be bases of a vector space V. Then there is a unique $n \times n$ matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$ so that

$$[\underline{x}]_{\mathcal{C}} = P_{\mathcal{C} \leftarrow \mathcal{B}} \, [\underline{x}]_{\mathcal{B}}.$$

The columns of $P_{\mathcal{C} \leftarrow \mathcal{B}}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$. That is,

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \left[\ [\underline{b}_1]_{\mathcal{C}} \ [\underline{b}_2]_{\mathcal{C}} \ \ldots \ [\underline{b}_n]_{\mathcal{C}}\ \right].$$

Proof.

$\square$

Example

Let

$$\underline{b}_1 = \begin{pmatrix} -9 \\ 1 \end{pmatrix}, \qquad \underline{b}_2 = \begin{pmatrix} -5 \\ -1 \end{pmatrix},$$

$$\underline{c}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}, \qquad \underline{c}_2 = \begin{pmatrix} 3 \\ -5 \end{pmatrix}.$$

Let $\mathcal{B} = \{\underline{b}_1, \underline{b}_2\}$ and let $\mathcal{C} = \{\underline{c}_1, \underline{c}_2\}$ be bases for $\mathbb{R}^2$. Find the change of coordinates matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ from $\mathcal{B}$ to $\mathcal{C}$.

Solution

By Theorem 15, $P_{\mathcal{C}\leftarrow\mathcal{B}} = [[\underline{b}_1]_{\mathcal{C}}, [\underline{b}_2]_{\mathcal{C}}]$.
So ,we need to find $[\underline{b}_1]_{\mathcal{C}}$ and $[\underline{b}_2]_{\mathcal{C}}$.
That is, we need to solve the vector equations:

$$\begin{aligned} \underline{b}_1 &= p_{11}\underline{c}_1 + p_{21}\underline{c}_2 = [\underline{c}_1, \underline{c}_2]\begin{pmatrix} p_{11} \\ p_{21} \end{pmatrix} \\ \underline{b}_2 &= p_{12}\underline{c}_1 + p_{22}\underline{c}_2 = [\underline{c}_1, \underline{c}_2]\begin{pmatrix} p_{12} \\ p_{22} \end{pmatrix} \end{aligned}$$

We will solve these simultaneously,

$$\begin{pmatrix} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & -9 & -5 \\ 0 & 7 & -35 & -21 \end{pmatrix} \rightarrow$$
$$\begin{pmatrix} 1 & 3 & -9 & -5 \\ 0 & 1 & -5 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & -9 & -5 \\ 0 & 1 & -5 & -3 \end{pmatrix} \rightarrow$$
$$\begin{pmatrix} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{pmatrix}$$

Solution continued ...

So $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} 6 & 4 \\ -5 & -3 \end{pmatrix}$.

Note

$$[c_1, c_2 | b_1, b_2] \to [I | P_{\mathcal{C}\leftarrow\mathcal{B}}].$$

Recall

Suppose that $\mathcal{B} = \{b_1, \ldots, b_n\}$ and $\mathcal{C} = \{c_1, \ldots, c_n\}$ bases for $\mathbb{R}^n$ and that $\vec{x} \in \mathbb{R}^n$.
Then we can write

$$\begin{aligned} P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} &= \vec{x} \qquad (P_{\mathcal{B}} = [b_1, \ldots b_n]), \\ P_{\mathcal{C}}[\vec{x}]_{\mathcal{C}} &= \vec{x} \qquad (P_{\mathcal{C}} = [c_1, \ldots c_n]). \end{aligned}$$

So,

$$[\vec{x}]_{\mathcal{C}} = P_{\mathcal{C}}^{-1}\vec{x} = P_{\mathcal{C}}^{-1}P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}.$$

So, $P_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{C}}^{-1}P_{\mathcal{B}}$.

Computational Note

$P_{\mathcal{C}\leftarrow\mathcal{B}}$ can be computed more quickly as in the previous example.