

MTHSC 3110 SECTION 4.7 – CHANGE OF BASIS

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EXAMPLE

Suppose that we have two bases $\mathcal{B} = \{\underline{b}_1, \underline{b}_2\}$ and $\mathcal{C} = \{\underline{c}_1, \underline{c}_2\}$ for a 2-dimensional vector space V . Since \mathcal{C} is a basis, we can express \mathcal{B} in terms of \mathcal{C} . Suppose that we know

$$\underline{b}_1 = 4\underline{c}_1 + 3\underline{c}_2 \quad \text{and} \quad \underline{b}_2 = 2\underline{c}_1 + \underline{c}_2.$$

Suppose $[\underline{x}]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

Find $[\underline{x}]_{\mathcal{C}}$.

THEOREM

Let $\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$ and $\mathcal{C} = \{\underline{c}_1, \dots, \underline{c}_n\}$ be bases of a vector space V . Then there is a unique $n \times n$ matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$ so that

$$[\underline{x}]_{\mathcal{C}} = P_{\mathcal{C} \leftarrow \mathcal{B}} [\underline{x}]_{\mathcal{B}}.$$

The columns of $P_{\mathcal{C} \leftarrow \mathcal{B}}$ are the \mathcal{C} -coordinate vectors of the vectors in the basis \mathcal{B} . That is,

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = [[\underline{b}_1]_{\mathcal{C}} \ [\underline{b}_2]_{\mathcal{C}} \ \dots \ [\underline{b}_n]_{\mathcal{C}}].$$

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PROOF.



EXAMPLE

Let

$$\begin{aligned}\underline{b}_1 &= \begin{pmatrix} -9 \\ 1 \end{pmatrix}, & \underline{b}_2 &= \begin{pmatrix} -5 \\ -1 \end{pmatrix}, \\ \underline{c}_1 &= \begin{pmatrix} 1 \\ -4 \end{pmatrix}, & \underline{c}_2 &= \begin{pmatrix} 3 \\ -5 \end{pmatrix}.\end{aligned}$$

Let $\mathcal{B} = \{\underline{b}_1, \underline{b}_2\}$ and let $\mathcal{C} = \{\underline{c}_1, \underline{c}_2\}$ be bases for \mathbb{R}^2 . Find the change of coordinates matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$ from \mathcal{B} to \mathcal{C} .

SOLUTION

By Theorem 15, $P_{\mathcal{C} \leftarrow \mathcal{B}} = [[\underline{b}_1]_{\mathcal{C}}, [\underline{b}_2]_{\mathcal{C}}]$.

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By Theorem 15, $P_{\mathcal{C} \leftarrow \mathcal{B}} = [[\underline{b}_1]_{\mathcal{C}}, [\underline{b}_2]_{\mathcal{C}}]$.

So ,we need to find $[\underline{b}_1]_{\mathcal{C}}$ and $[\underline{b}_2]_{\mathcal{C}}$.

SOLUTION

By Theorem 15, $P_{C \leftarrow B} = [[\underline{b}_1]_C, [\underline{b}_2]_C]$.

So, we need to find $[\underline{b}_1]_C$ and $[\underline{b}_2]_C$.

That is, we need to solve the vector equations:

$$\underline{b}_1 = p_{11}\underline{c}_1 + p_{21}\underline{c}_2 =$$

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$$\underline{b}_1 = p_{11}\underline{c}_1 + p_{21}\underline{c}_2 = [\underline{c}_1, \underline{c}_2] \begin{pmatrix} p_{11} \\ p_{21} \end{pmatrix}$$

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$$\underline{b}_2 = p_{12}\underline{c}_1 + p_{22}\underline{c}_2 =$$

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By Theorem 15, $P_{\mathcal{C} \leftarrow \mathcal{B}} = [[\underline{b}_1]_{\mathcal{C}}, [\underline{b}_2]_{\mathcal{C}}]$.

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We will solve these simultaneously,

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We will solve these simultaneously,

$$\begin{pmatrix} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & -9 & -5 \\ 0 & 7 & -35 & -21 \end{pmatrix} \rightarrow$$
$$\begin{pmatrix} 1 & 3 & -9 & -5 \\ 0 & 1 & -5 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & -9 & -5 \\ 0 & 1 & -5 & -3 \end{pmatrix} \rightarrow$$
$$\begin{pmatrix} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{pmatrix}$$

SOLUTION CONTINUED ...

$$\text{So } P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} 6 & 4 \\ -5 & -3 \end{pmatrix}.$$

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NOTE

$$[c_1, c_2 | b_1, b_2] \rightarrow [I | P_{\mathcal{C} \leftarrow \mathcal{B}}].$$

RECALL

Suppose that $\mathcal{B} = \{b_1, \dots, b_n\}$ and $\mathcal{C} = \{c_1, \dots, c_n\}$ bases for \mathbb{R}^n and that $\vec{x} \in \mathbb{R}^n$.

RECALL

Suppose that $\mathcal{B} = \{b_1, \dots, b_n\}$ and $\mathcal{C} = \{c_1, \dots, c_n\}$ bases for \mathbb{R}^n and that $\vec{x} \in \mathbb{R}^n$.

Then we can write

$$P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = \vec{x} \quad (P_{\mathcal{B}} = [b_1, \dots, b_n]),$$

$$P_{\mathcal{C}}[\vec{x}]_{\mathcal{C}} = \vec{x} \quad (P_{\mathcal{C}} = [c_1, \dots, c_n]).$$

So,

$$[\vec{x}]_{\mathcal{C}} =$$

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So,

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So,

$$[\vec{x}]_{\mathcal{C}} = P_{\mathcal{C}}^{-1}\vec{x} = P_{\mathcal{C}}^{-1}P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}.$$

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$$\begin{aligned} P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} &= \vec{x} & (P_{\mathcal{B}} &= [b_1, \dots, b_n]), \\ P_{\mathcal{C}}[\vec{x}]_{\mathcal{C}} &= \vec{x} & (P_{\mathcal{C}} &= [c_1, \dots, c_n]). \end{aligned}$$

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$$[\vec{x}]_{\mathcal{C}} = P_{\mathcal{C}}^{-1}\vec{x} = P_{\mathcal{C}}^{-1}P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}.$$

$$\text{So, } P_{\mathcal{C} \leftarrow \mathcal{B}} = P_{\mathcal{C}}^{-1}P_{\mathcal{B}}.$$

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Then we can write

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$$[\vec{x}]_{\mathcal{C}} = P_{\mathcal{C}}^{-1}\vec{x} = P_{\mathcal{C}}^{-1}P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}.$$

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COMPUTATIONAL NOTE

$P_{\mathcal{C} \leftarrow \mathcal{B}}$ can be computed more quickly as in the previous example.