MTHSC 3110 Section 5.1 – Eigenvalues and Eigenvectors

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Definition

Let A be an $n \times n$ matrix. A *non-zero* vector $\vec{x} \in \mathbb{R}^n$ is called an eigenvector of A if there exists some scalar $\lambda \in \mathbb{R}$ so that $A\vec{x} = \lambda\vec{x}$. If $\vec{x}$ is an eigenvector of A, the corresponding value λ is called an eigenvalue of A, and we say that λ is an eigenvalue of A with eigenvector $\vec{x}$.

Note

While an eigenvector $\vec{x}$ must be non-zero (so that we are always excluding the trivial case $A\vec{0} = \vec{0}$), it is possible for the value λ to be zero.

Example

$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ has eigenvector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ with eigenvalue 0.

Note

If λ is an eigenvalue for A, the eigenvectors for A corresponding to λ along with $\vec{0}$ form a subspace of $\mathbb{R}^n$.

Example

$$\begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ -5 \end{pmatrix} = \begin{pmatrix} -24 \\ 20 \end{pmatrix} = -4 \begin{pmatrix} 6 \\ -5 \end{pmatrix}.$$

Thus we see that $\begin{pmatrix} 6 \\ -5 \end{pmatrix}$ is an eigenvector of this matrix, and -4 is the corresponding eigenvalue.

Example

Show that 7 is an eigenvalue for

$$A = \begin{pmatrix} 2 & 4 \\ 5 & 3 \end{pmatrix}.$$

Note

For any scalar λ,

$$\begin{aligned} A\vec{x} &= \lambda\vec{x} \\ &\Leftrightarrow A\vec{x} - \lambda\vec{x} = \vec{0} \\ &\Leftrightarrow A\vec{x} - \lambda I\vec{x} = \vec{0} \\ &\Leftrightarrow (A - \lambda I)\,\vec{x} = \vec{0} \\ &\Leftrightarrow \vec{x} \in Nul(A - \lambda I) \end{aligned}$$

Definition

If $\dim(\mathrm{Nul}(A - \lambda I)) > 0$, then $\mathrm{Nul}(A - \lambda I)$ is called the eigenspace for A corresponding to the eigenvalue λ, since any $\vec{x} \in \mathrm{Nul}(A - \lambda I)$ satisfies $A\vec{x} = \lambda\vec{x}$.

Example

Find a basis for the eigenspace corresponding to the eigenvector 2 for the matrix

$$A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}.$$

Theorem

The eigenvalues of an upper triangular matrix (or of a lower triangular matrix) are its diagonal entries.

Invertible Matrix Theorem

The number 0 is an eigenvalue of A

$\iff$ there exists $\vec{x} \neq \vec{0}$ so that $A\vec{x} = 0\vec{x}$

$\iff$ there exists $\vec{x} \neq \vec{0}$ so that $A\vec{x} - 0\vec{x} = \vec{0}$.

$\iff$ There exists $\vec{x} \neq \vec{0} \in \mathrm{Nul}(A)$.

$\iff$ A is not invertible.

Theorem

If $\vec{v}_1, \dots, \vec{v}_r$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \dots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\vec{v}_1, \dots, \vec{v}_r\}$ is linearly independent.

Example

In many applications, one is interested in studying repeated application of a linear map.
Suppose we would like to study the long term behavior of a sequence $\{\vec{x}_k\}$ satisfying $\vec{x}_{k+1} = A\vec{x}_k$.
Describe the long term behavior of such a sequence where $\vec{x}_0$ is an eigenvector of A with eigenvalue λ.