MTHSC 3110 Section 5.1 – Eigenvalues and Eigenvectors

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Definition

Let A be an $n \times n$ matrix. A *non-zero* vector $\vec{x} \in \mathbb{R}^n$ is called an eigenvector of A if there exists some scalar $\lambda \in \mathbb{R}$ so that $A\vec{x} = \lambda\vec{x}$.

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Example

$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ has eigenvector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ with eigenvalue 0.

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If λ is an eigenvalue for A, the eigenvectors for A corresponding to λ along with $\vec{0}$ form a subspace of $\mathbb{R}^n$.

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$$\begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ -5 \end{pmatrix} = \begin{pmatrix} -24 \\ 20 \end{pmatrix} = -4 \begin{pmatrix} 6 \\ -5 \end{pmatrix}.$$

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Thus we see that $\begin{pmatrix} 6 \\ -5 \end{pmatrix}$ is an eigenvector of this matrix, and -4 is the corresponding eigenvalue.

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If $\dim(\mathrm{Nul}(A - \lambda I)) > 0$, then $\mathrm{Nul}(A - \lambda I)$ is called the <u>eigenspace</u> for A corresponding to the eigenvalue λ, since any $\vec{x} \in \mathrm{Nul}(A - \lambda I)$ satisfies $A\vec{x} = \lambda\vec{x}$.

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Example

Find a basis for the eigenspace corresponding to the eigenvector 2 for the matrix

$$A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}.$$

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Theorem

If $\vec{v}_1, \ldots, \vec{v}_r$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\vec{v}_1, \ldots, \vec{v}_r\}$ is linearly independent.

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