MTHSC 3110 Section 5.2 – The Characteristic Equation

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Note

To find the eigenvalues of a matrix A, we must determine values $\lambda \in \mathbb{R}$ such that $\text{Nul}(A - \lambda I) \neq \{\vec{0}\}$.
Recall that this is equivalent to $A - \lambda I$ being singular or $\det(A - \lambda I) = 0$.

Example

Find the eigenvalues of $A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$

Fact

For a general 2×2 *matrix* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$*, the formula for the determinant enables us to compute the eigenvalues easily.*

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} (a-\lambda) & b \\ c & (d-\lambda) \end{pmatrix} \\ &= (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - (a+d)\lambda + (ad - bc). \end{aligned}$$

So that the quadratic formula gives

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}.$$

Advanced Example

Let $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ be the matrix which rotates a vector counter clockwise through an angle θ. Then we have

$$\det(A - \lambda I) = \lambda^2 - 2\lambda\cos\theta + 1.$$

So $\lambda = \cos\theta \pm \sqrt{cos^2\theta - 1} = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$, where $i = \sqrt{-1}$. So unless $\sin\theta = 0$, there are no real eigenvalues.

Theorem (The Invertible Matrix Theorem continued)

Let A be an $n \times n$ matrix. Then A is invertible if and only if

(s) *The number 0 is* not *an eigenvalue of A.*

(t) *The determinant of A is not zero.*

Theorem (Properties of Determinants)

Let A and B be $n \times n$ matrices.

(a) *A is invertible if and only if* $\det(A) \neq 0$

(b) $\det(AB) = \det(A)\det(B)$.

(c) $\det(A^T) = \det(A)$.

(d) *If A is triangular, then* $\det(A)$ *is the product of the entries on the main diagonal.*

(e) *A row replacement operation on A doesn't change the determinant. A row interchange switches the sign of the determinant. A row scaling also scales the determinant by the same scale factor.*

Theorem

A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation

$$\det(A - \lambda I) = 0.$$

Fact

$\det(A - \lambda I)$ *is a polynomial in λ of degree n.*
Hence, an $n \times n$ matrix has at most n eigenvalues.

Definition

1. The degree n polynomial $\det(A - \lambda I)$ is called the characteristic polynomial.
2. The (algebraic) multiplicity of the eigenvalue λ_0 is the power of $(\lambda - \lambda_0)$ appearing in the factorization of the characteristic polynomial.

Example

Find the eigenvalues and their multiplicities of

$$A = \begin{pmatrix} 6 & 5 & 0 & -5 \\ 0 & -3 & 1 & 2 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & 0 & 7 \end{pmatrix}.$$

Definition

Two $n \times n$ matrices A and B are similar if there is an invertible matrix P so that $P^{-1}AP = B$, or equivalently $A = PBP^{-1}$.

Theorem

If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomials.

Example

Let $A = \begin{pmatrix} .95 & .03 \\ .05 & .97 \end{pmatrix}$.

Take $x_0 = \begin{pmatrix} .6 \\ .4 \end{pmatrix}$, and $x_{k+1} = Ax_k$ for $k \geq 0$.

Analyze the long term behavior of this sequence.