MTHSC 3110 Section 5.2 – The Characteristic Equation

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Example

Find the eigenvalues of $A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$

Fact

For a general 2×2 *matrix* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, *the formula for the determinant enables us to compute the eigenvalues easily.*

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So that the quadratic formula gives

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}.$$

Advanced Example

Let $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ be the matrix which rotates a vector counter clockwise through an angle θ. Then we have

$$\det(A - \lambda I) = \lambda^2 - 2\lambda\cos\theta + 1.$$

So $\lambda = \cos\theta \pm \sqrt{cos^2\theta - 1} = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$, where $i = \sqrt{-1}$. So unless $\sin\theta = 0$, there are no real eigenvalues.

Theorem (The Invertible Matrix Theorem continued)

Let A be an $n \times n$ matrix. Then A is invertible if and only if

(s) *The number 0 is* not *an eigenvalue of A.*

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Theorem (Properties of Determinants)

Let A and B be $n \times n$ matrices.

(A) *A is invertible if and only if $\det(A) \neq 0$*

(B) $\det(AB) = \det(A)\det(B)$.

(C) $\det(A^T) = \det(A)$.

(D) *If A is triangular, then $\det(A)$ is the product of the entries on the main diagonal.*

(E) *A row replacement operation on A doesn't change the determinant. A row interchange switches the sign of the determinant. A row scaling also scales the determinant by the same scale factor.*

Theorem

A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation

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$\det(A - \lambda I)$ *is a polynomial in λ of degree n.*
Hence, an $n \times n$ matrix has at most n eigenvalues.

Definition

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Example

Find the eigenvalues and their multiplicities of

$$A = \begin{pmatrix} 6 & 5 & 0 & -5 \\ 0 & -3 & 1 & 2 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & 0 & 7 \end{pmatrix}.$$

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Let $A = \begin{pmatrix} .95 & .03 \\ .05 & .97 \end{pmatrix}$.

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Analyze the long term behavior of this sequence.