MTHSC 3110 Section 5.3 – Diagonalization

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Definition

A matrix $D = (d_{ij})$ if diagonal if $d_{ij} = 0$ whenever $i \neq j$.

Example

Let $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.
Compute D^k for $k = 1, 2, \ldots$.

Example

Suppose that P is invertible and that $A = PDP^{-1}$. Compute A^k for $k = 1, 2, \ldots$.

Definition

An $n \times n$ matrix A is said to be diagonalizable if there exists a an invertible matrix P and a diagonal matrix D (each $n \times n$) with $A = PDP^{-1}$ (or equivalently, since P is invertible, $AP = PD$). That is, A is diagonalizable if it is similar to a diagonal matrix.

Theorem (The Diagonalization Theorem)

An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.
In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case the diagonal entries of D are the eigenvalues of A, with the jj^{th} entry of D being the eigenvalue corresponding to the eigenvector which is the j^{th} column of P.

Note

Suppose A is $n \times n$. Then A is diagonalizable if and only if there is a basis for $\mathbb{R}^n$ made up of eigenvectors for A.

Proof

Suppose that D is a diagonal $n \times n$ matrix with entries $\lambda_1, \ldots, \lambda_n$, and $P = [\vec{p}_1, \ldots, \vec{p}_n]$ is $n \times n$.

Then,

$$PD = [P\vec{d}_1, \ldots, P\vec{d}_n] = [\lambda_1 \vec{p}_1, \ldots, \lambda_n \vec{p}_n] .$$

Also,

$$AP = [A\vec{p}_1, \ldots, A\vec{p}_n]$$

So $AP = PD$ if and only if $A\vec{p}_i = \lambda_i \vec{p}_i$ for $1 \leq i \leq n$.

Proof continued...

If A is diagonalizable, then $A = PDP^{-1} \Rightarrow AP = PD$.
From our earlier observation, it follows that $\vec{p_i}$ is an eigenvector for A w.r.t the eigenvalue λ_i.
Further, since P is invertible, it follows that $\{\vec{p_1}, \dots, \vec{p_n}\}$ is a basis for $\mathbb{R}^n$.
Now suppose that $\{\vec{p_1}, \dots, \vec{p_n}\}$ is a basis for $\mathbb{R}^n$ composed of eigenvectors for A with eigenvalues $\lambda_1, \dots, \lambda_n$.
Let $P = [\vec{p_1}, \dots, \vec{p_n}]$ and let D be the diagonal $n \times n$ matrix with entries $\lambda_1, \dots, \lambda_n$.
Then from before, we have $AP = PD$. Also, P is invertible.
Thus, $A = PDP^{-1}$ and A is diagonalizable. $\square$

Example

Is $A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$ diagonalizable? If so, find P and D.

Example

Is $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ diagonalizable? If so, find P and D.

Corollary

An $n \times n$ matrix A having n distinct eigenvalues is diagonalizable.

Proof.

We saw earlier that eigenvectors corresponding to distinct eigenvalues are linearly independent. Each eigenvalue has at least one corresponding eigenvector. Hence, we have n linearly independent eigenvectors, and A is diagonalizable. $\square$

Theorem

Let A be $n \times n$ with distinct eigenvalues $\lambda_1, \ldots, \lambda_p$.

1. *For $1 \leq k \leq p$, the dimension of the eigenspace w.r.t. λ_k is less than or equal to the algebraic multiplicity of λ_k. (-i.e. $\dim(Nul(A - \lambda_k I)) \leq mult(\lambda_k)$).*
2. *A is diagonalizable if and only if*

$$\sum_{k=1}^{p} \dim \quad (Nul(A - \lambda_k I)) = n$$

$$\Leftrightarrow \quad \dim(Nul(A - \lambda_k I)) = mult(\lambda_k) \qquad \text{for } 1 \leq k \leq p.$$

3. *If A is diagonalizable and if $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p$ are bases for the eigenspaces for A corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_p$ respectively, then $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_p$ is an eigenvector basis for $\mathbb{R}^n$.*