# MTHSC 3110 Section 5.3 – Diagonalization

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Suppose that $P$ is invertible and that $A = PDP^{-1}$. Compute $A^k$ for $k = 1, 2, \ldots$.

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An $n \times n$ matrix $A$ is said to be diagonalizable if there exists a an invertible matrix $P$ and a diagonal matrix $D$ (each $n \times n$) with $A = PDP^{-1}$ (or equivalently, since $P$ is invertible, $AP = PD$). That is, $A$ is diagonalizable if it is similar to a diagonal matrix.

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*An $n \times n$ matrix $A$ is diagonalizable if and only if it has n linearly independent eigenvectors.*
*In fact, $A = PDP^{-1}$, with $D$ a diagonal matrix, if and only if the columns of $P$ are n linearly independent eigenvectors of $A$. In this case the diagonal entries of $D$ are the eigenvalues of $A$, with the $jj^{th}$ entry of $D$ being the eigenvalue corresponding to the eigenvector which is the $j^{th}$ column of $P$.*

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Also,

$$AP = [A\vec{p}_1, \ldots, A\vec{p}_n]$$

So $AP = PD$ if and only if $A\vec{p}_i = \lambda_i\vec{p}_i$ for $1 \leq i \leq n$.

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Now suppose that $\{\vec{p_1}, \dots, \vec{p_n}\}$ is a basis for $\mathbb{R}^n$ composed of eigenvectors for $A$ with eigenvalues $\lambda_1, \dots, \lambda_n$.

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Let $P = [\vec{p_1}, \dots, \vec{p_n}]$ and let $D$ be the diagonal $n \times n$ matrix with entries $\lambda_1, \dots, \lambda_n$.

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Now suppose that $\{\vec{p_1}, \ldots, \vec{p_n}\}$ is a basis for $\mathbb{R}^n$ composed of eigenvectors for $A$ with eigenvalues $\lambda_1, \ldots, \lambda_n$.
Let $P = [\vec{p_1}, \ldots, \vec{p_n}]$ and let $D$ be the diagonal $n \times n$ matrix with entries $\lambda_1, \ldots, \lambda_n$.
Then from before, we have $AP = PD$. Also, $P$ is invertible.
Thus, $A = PDP^{-1}$ and $A$ is diagonalizable. $\square$

**Example**

Is $A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$ diagonalizable? If so, find $P$ and $D$.

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Is $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ diagonalizable? If so, find $P$ and $D$.

## Corollary

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**Proof.**

We saw earlier that eigenvectors corresponding to distinct eigenvalues are linearly independent. Each eigenvalue has at least one corresponding eigenvector. Hence, we have $n$ linearly independent eigenvectors, and $A$ is diagonalizable. $\square$

## Theorem

*Let $A$ be $n \times n$ with distinct eigenvalues $\lambda_1, \ldots, \lambda_p$.*

1. *For $1 \leq k \leq p$, the dimension of the eigenspace w.r.t. $\lambda_k$ is less than or equal to the algebraic multiplicity of $\lambda_k$. (-i.e. $\dim(Nul(A - \lambda_k I)) \leq mult(\lambda_k)$ ).*
2. *$A$ is diagonalizable if and only if*

$$\sum_{k=1}^{p} \dim \quad (Nul(A - \lambda_k I)) = n$$

$$\Leftrightarrow \quad \dim(Nul(A - \lambda_k I)) = mult(\lambda_k) \qquad \text{for } 1 \leq k \leq p.$$

3. *If $A$ is diagonalizable and if $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p$ are bases for the eigenspaces for $A$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_p$ respectively, then $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_p$ is an eigenvector basis for $\mathbb{R}^n$.*