MTHSC 3110 Section 6.1 – Inner Product, Length and Orthoganality

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Definition

Suppose that $\vec{u}, \vec{v} \in \mathbb{R}^n$. We define the inner product or dot procuct or $\vec{u}$ and $\vec{v}$ as

$$u \cdot v = u^t v = \sum_{i=1}^{n} u_i v_i.$$

Example

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} = (1)(-1) + (2)(-2) + (3)(1) = -2.$$

Theorem

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and let c be a scalar. Then

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.
2. $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$.
3. $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$.
4. $\vec{u} \cdot \vec{u} \geq 0$ *and* $\vec{u} \cdot \vec{u} = 0$ *if and only if* $\vec{u} = \vec{0}$.
5. $(c_1\vec{u}_1 + \cdots + c_p\vec{u}_p) \cdot \vec{w} = c_1\vec{u}_1 \cdot \vec{w} + c_p\vec{u}_p \cdot \vec{w}$.

Definition (Length)

The *length* (or *norm*) of $\vec{v}$ is the non-negative scalar $\|\vec{v}\|$ defined by

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \quad \text{and} \quad \|\vec{v}\|^2 = \vec{v} \cdot \vec{v}.$$

Note

This definition is chosen so that the Pythagorean theorem holds (that is, in two dimensions the length c of the vector which is the hypotenuse of a right triangle with horizontal length a and vertical height b satisfies $a^2 + b^2 = c^2$.

Fact

1. *For any scalar c,* $\|c\vec{v}\| = |c|\ \|\vec{v}\|$.
2. *A vector of length 1 is called a unit vector.*
3. *If* $\vec{v} \neq \vec{0}$ *then* $\frac{1}{\|\vec{v}\|}\vec{v}$ *is a unit vector and is in the same direction as* $\vec{v}$.
4. *The above process is called normalizing.*

Example

Find a unit vector which is in the same direction as $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

Definition

For $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$, the *distance between $\vec{u}$ and $\vec{v}$*, written as $\text{dist}(\vec{u}, \vec{v})$ is the length of the vector $\vec{u} - \vec{v}$. That is,

$$\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|.$$

Example

Compute the distance $\text{dist}\left(\begin{pmatrix} 7 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \end{pmatrix}\right)$.

Definition

Two vectors in $\mathbb{R}^n$ are *orthogonal* if and only if $\vec{u} \cdot \vec{v} = 0$.

Note

$$\begin{aligned} \|\vec{u} - \vec{v}\|^2 &= (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) \\ &= \vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} + \vec{v}\vec{v} \\ &= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\vec{u} \cdot \vec{v} \end{aligned}$$

Theorem (Pythagorus)

$||\vec{u} - \vec{v}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2$ *if and only if* $\vec{u}$ *and* $\vec{v}$ *are orthogonal.*

Definition

Suppose that $W \leq \mathbb{R}^n$.

1. If for all $\vec{w} \in W$, $\vec{z} \cdot \vec{w} = 0$,l then we say that $\vec{z}$ is orthogonal to W and write $z \perp W$.
2. We define the orthogonal compliment of W as $W^{\perp} = \{\vec{z} \in \mathbb{R}^n \mid \vec{z} \perp W\}$.

Example

Suppose that $W = \text{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right)$.

Describe $W^{\perp}$.

Fact

1. $\vec{x} \in W^{\perp}$ if and only if $\vec{x} \perp \vec{w}$ for all $\vec{w}$ in a spanning set for W.
2. $W^{\perp} \leq \mathbb{R}^n$.

Theorem

Suppose that A is an $n \times n$ matrix.

1. $(Row(A))^{\perp} = Nul(A)$.
2. $(Col(A))^{\perp} = Nul(A^t)$.

Note

In two or three dimensions, the projection of $\vec{v}$ onto $\vec{u}$ has length $\|\vec{v}\| \cos\theta$, where θ is the angle between the vectors. Hence

$$\|\vec{v}\| \cos\theta = c\|\vec{u}\|$$

so that

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta.$$

In higher dimensions than three we use this to *define* the angle between two vectors.