MTHSC 3110 Section 6.3 – Orthogonal Projection

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We extend the idea of projections onto a 1-dimensional space to the idea of projecting onto an arbitrary subspace.

Theorem

Let $W < \mathbb{R}^n$. Then any vector $\vec{y} \in \mathbb{R}^n$ can be written uniquely as

$$\vec{y} = \hat{y} + \vec{z}$$

where $\hat{y} \in W$, and $\vec{z} \in W^{\perp}$.
If $S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ is an orthogonal basis for W, then $\hat{y}$ is given in terms of the basis S by

$$\hat{y} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_k\vec{u}_k$$

then

$$c_j = \frac{\vec{y} \cdot \vec{u}_j}{\vec{u}_j \cdot \vec{u}_j}$$

and $\vec{z} = \vec{y} - \hat{y}$.

Example

Let $\vec{u}_1 = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Note that $\vec{u}_1 \cdot \vec{u}_2 = 0$. Let $W = \mathrm{Span}(\vec{u}_1, \vec{u}_2)$.

Find $\hat{y}$ and $\vec{z}$ so that $\vec{y} = \hat{y} + \vec{z}$, $\hat{y} \in W$ and $\vec{z} \in W^{\perp}$.

Notation

$\hat{y} = \mathrm{Proj}_W(\vec{y})$. is called the orthogonal projection of $\vec{y}$ onto W.

Theorem (Best Approximation Theorem)

Let W be a subspace of $\mathbb{R}^n$, $\vec{y} \in \mathbb{R}^n$ and $\hat{y} = Proj_W(\vec{y})$. Then $\hat{y}$ is the closest point in W to $\vec{y}$. That is, for every $\vec{v} \in W$, if $\vec{v} \neq \hat{y}$ then

$$\|\vec{y} - \hat{y}\| < \|\vec{y} - \vec{v}\|.$$

Proof.

Take $\vec{v} \in W$ with $\vec{v} \neq \hat{y}$.
Since $\vec{v}, \hat{y} \in W$, we have $\vec{0} \neq (\hat{y} - \vec{v}) \in W$.
Since $\vec{y} - \hat{y}$ is in $W^{\perp}$, $(\vec{y} - \hat{y}) \cdot (\hat{y} - \vec{v}) = 0$.
Now, $\vec{y} - \vec{v} = (\vec{y} - \hat{y}) + (\hat{y} - \vec{v})$, and so the Pythagorean theorem implies

$$\|\vec{y} - \vec{v}\|^2 = \|\vec{y} - \hat{y}\|^2 + \|\hat{y} - \vec{v}\|^2 > \|\vec{y} - \hat{y}\|^2.$$

$\square$

Note

We define the distance from $\vec{y}$ to a subspace W by

$$\text{dist}(\vec{y}, W) = \text{dist}(\vec{y}, \text{Proj}_W(\vec{y})).$$

Example

Let $\vec{u}_1 = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} -1 \\ -5 \\ 10 \end{pmatrix}$, $W = \text{Span}(\vec{u}_1, \vec{u}_2)$. Compute $\text{dist}(\vec{y}, W)$.

Theorem

If $\{\vec{u}_1, \dots, \vec{u}_p\}$ is an orthonormal basis for W, then

$$Proj_W(\vec{y}) = (\vec{y} \cdot \vec{u}_1)\vec{u}_1 + (\vec{y} \cdot \vec{u}_2)\vec{u}_2 + \dots (\vec{y} \cdot \vec{u}_p)\vec{u}_p.$$

Furthermore, if $U = [\vec{u}_1 \;\; \vec{u}_2 \;\; \dots \;\; \vec{u}_p]$ then for every $\vec{y} \in \mathbb{R}^n$,

$$Proj_W(\vec{y}) = UU^T\vec{y}$$