MTHSC 3110 Section 6.3 – Orthogonal Projection

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Theorem

Let $W < \mathbb{R}^n$. Then any vector $\vec{y} \in \mathbb{R}^n$ can be written uniquely as

$$\vec{y} = \hat{y} + \vec{z}$$

where $\hat{y} \in W$, and $\vec{z} \in W^\perp$.

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where $\hat{y} \in W$, and $\vec{z} \in W^{\perp}$.
If $S = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k\}$ is an orthogonal basis for W, then $\hat{y}$ is given in terms of the basis S by

$$\hat{y} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + \cdots + c_k \vec{u}_k$$

then

$$c_j = \frac{\vec{y} \cdot \vec{u}_j}{\vec{u}_j \cdot \vec{u}_j}$$

and $\vec{z} = \vec{y} - \hat{y}$.

Example

Let $\vec{u}_1 = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Note that $\vec{u}_1 \cdot \vec{u}_2 = 0$. Let $W = \text{Span}(\vec{u}_1, \vec{u}_2)$.

Find $\hat{y}$ and $\vec{z}$ so that $\vec{y} = \hat{y} + \vec{z}$, $\hat{y} \in W$ and $\vec{z} \in W^{\perp}$.

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Theorem (Best Approximation Theorem)

Let W be a subspace of $\mathbb{R}^n$, $\vec{y} \in \mathbb{R}^n$ and $\hat{y} = Proj_W(\vec{y})$. Then $\hat{y}$ is the closest point in W to $\vec{y}$.

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Theorem (Best Approximation Theorem)

Let W be a subspace of $\mathbb{R}^n$, $\vec{y} \in \mathbb{R}^n$ and $\hat{y} = \text{Proj}_W(\vec{y})$. Then $\hat{y}$ is the closest point in W to $\vec{y}$. That is, for every $\vec{v} \in W$, if $\vec{v} \neq \hat{y}$ then

$$\|\vec{y} - \hat{y}\| < \|\vec{y} - \vec{v}\|.$$

Proof.

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Note

We define the distance from $\vec{y}$ to a subspace W by

$$\text{dist}(\vec{y}, W) = \text{dist}(\vec{y}, \text{Proj}_W(\vec{y})).$$

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$W = \text{Span}(\vec{u}_1, \vec{u}_2)$. Compute $\text{dist}(\vec{y}, W)$.

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Theorem

If $\{\vec{u}_1, \dots, \vec{u}_p\}$ is an orthonormal basis for W, then

$$Proj_W(\vec{y}) = (\vec{y} \cdot \vec{u}_1)\vec{u}_1 + (\vec{y} \cdot \vec{u}_2)\vec{u}_2 + \dots (\vec{y} \cdot \vec{u}_p)\vec{u}_p.$$

Furthermore, if $U = [\vec{u}_1 \ \vec{u}_2 \ \dots \ \vec{u}_p]$ then for every $\vec{y} \in \mathbb{R}^n$,

$$Proj_W(\vec{y}) = UU^T\vec{y}$$