MTHSC 3110 Section 6.4 – The Gram-Schmidt Process

Kevin James

Overview

The Gram-Schmidt algorithm constructs an orthogonal or orthonormal basis for any subspace $\{\vec{0}\} \neq W \leq \mathbb{R}^n$ starting with any basis for W.

Example

Let $W = \mathrm{Span}(\vec{x}_1, \vec{x}_2)$ where $\vec{x}_1 = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}$ and $\vec{x}_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$. Find an orthogonal basis for W.

Now find an orthonormal basis for W.

Example

Let $W = \text{Span}(\vec{x}_1, \vec{x}_2, \vec{x}_3)$ where $\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ and $\vec{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$. Clearly the vectors are linearly independent, so W is a three-dimensional subspace of $\mathbb{R}^4$. Find an orthogonal basis for W.

Now find an orthonormal basis for W.

Theorem

Suppose $W \leq \mathbb{R}^n$ *has a basis* $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_p\}$*. Then,*

$$\begin{aligned}
\vec{v}_1 &= \vec{x}_1 \\
\vec{v}_2 &= \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 \\
\vec{v}_3 &= \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 \\
&\vdots \\
\vec{v}_p &= \vec{x}_p - \frac{\vec{x}_p \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \cdots - \frac{\vec{x}_p \cdot \vec{v}_{p-1}}{\vec{v}_{p-1} \cdot \vec{v}_{p-1}}\vec{v}_{p-1}
\end{aligned}$$

is an orthogonal basis for W*.*
Also,

$$Span(\vec{v}_1, \dots, \vec{v}_k) = Span(\vec{x}_1, \dots, \vec{x}_k)$$

for $1 \leq k \leq p$*.*

Theorem (QR Factorization)

Suppose that A is a $m \times n$ matrix with linearly independent columns. Then there exist matrices Q and R so that

1. $A = QR$.
2. *Q is $m \times n$ and the columns of Q form an orthonormal basis for $Col(A)$, the column space of A.*
3. *R is an upper triangular, square matrix with positive entries on the diagonal.*

Proof Sketch

The columns of A are linearly independent, so they form a basis for $\mathrm{Col}(A)$. Convert them to an orthonormal basis via the Gram Schmidt algorithm. Check that $A = QR$ with Q and R as claimed follows from the way we construct the orthonormal basis.