# MTHSC 3110 Section 6.4 – The Gram-Schmidt Process

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## Overview

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Let $W = \text{Span}(\vec{x}_1, \vec{x}_2)$ where $\vec{x}_1 = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}$ and $\vec{x}_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$. Find an orthogonal basis for $W$.

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Now find an orthonormal basis for $W$.

## Example

Let $W = \text{Span}(\vec{x}_1, \vec{x}_2, \vec{x}_3)$ where $\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ and $\vec{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$. Clearly the vectors are linearly independent, so $W$ is a three-dimensional subspace of $\mathbb{R}^4$. Find an orthogonal basis for $W$.

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## Theorem

*Suppose* $W \leq \mathbb{R}^n$ *has a basis* $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_p\}$. *Then,*

$$\begin{aligned}
\vec{v}_1 &= \vec{x}_1 \\
\vec{v}_2 &= \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 \\
\vec{v}_3 &= \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 \\
&\vdots \\
\vec{v}_p &= \vec{x}_p - \frac{\vec{x}_p \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \cdots - \frac{\vec{x}_p \cdot \vec{v}_{p-1}}{\vec{v}_{p-1} \cdot \vec{v}_{p-1}}\vec{v}_{p-1}
\end{aligned}$$

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\end{aligned}$$

*is an orthogonal basis for* $W$.
*Also,*

$$Span(\vec{v}_1, \ldots, \vec{v}_k) = Span(\vec{x}_1, \ldots, \vec{x}_k)$$

*for* $1 \leq k \leq p$.

### Theorem (QR Factorization)

*Suppose that $A$ is a $m \times n$ matrix with linearly independent columns. Then there exist matrices $Q$ and $R$ so that*

1. $A = QR$.
2. *$Q$ is $m \times n$ and the columns of $Q$ form an orthonormal basis for $Col(A)$, the column space of $A$.*
3. *$R$ is an upper triangular, square matrix with positive entries on the diagonal.*

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## Proof Sketch

The columns of $A$ are linearly independent, so they form a basis for $\mathrm{Col}(A)$. Convert them to an orthonormal basis via the Gram Schmidt algorithm. Check that $A = QR$ with $Q$ and $R$ as claimed follows from the way we construct the orthonormal basis.