MTHSC 3110 Section 6.5 – Least Squares Solutions

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Definition

Suppose that A is an $m \times n$ matrix. A least squares solution of the matrix equation $A\vec{x} = \vec{b}$ is a vector $\hat{x} \in \mathbb{R}^n$ satisfying

$$||\vec{b} - A\hat{x}|| \leq ||\vec{b} - A\vec{x}||$$

for all $\vec{x} \in \mathbb{R}^n$.

Example

Find a least squares solution to $A\vec{x} = \vec{b}$ where

$$A = \begin{pmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{pmatrix}. \text{ and } \vec{b} = \begin{pmatrix} -1 \\ 2 \\ 1 \\ 6 \end{pmatrix}.$$

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From our previous work, a solution is given by $\hat{x} = \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix}$.

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The set of least squares solutions to $A\vec{x} = \vec{b}$ is the nonempty set of solutions of

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Theorem

The matrix $A^T A$ is invertible if and only if the columns of A are linearly independent. In this case, the equation $A\vec{x} = \vec{b}$ has a unique least squares solution, and it is given by

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Theorem

Given an $m \times n$ matrix A with linearly independent columns, let $A = QR$ be a QR-factorization of A. Then for each $\vec{b} \in \mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a unique least-squares solution, given by

$$\hat{x} = R^{-1} Q^T \vec{b}.$$