MTHS 3190 Binary Operations

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Definition

A binary operation on a nonempty set A is a mapping f form $A \times A$ to A. That is $f \subseteq A \times A \times A$ and f has the property that for each $(a, b) \in A \times A$, there is precisely one $c \in A$ such that $(a, b, c) \in f$.

Notation

If f is a binary operation on A and if $(a, b, c) \in f$ then we have already seen the notation $f(a, b) = c$. For binary operations, it is customary to write instead

$$a \ f \ b = c,$$

or perhaps

$$a * b = c.$$

Example

Some binary operations on $\mathbb{Z}$ are

1. $x * y = x + y$
2. $x * y = x - y$
3. $x * y = xy$
4. $x * y = x + 2y + 3$
5. $x * y = 1 + xy$

Definition

Suppose that $*$ is a binary operation of a nonempty set A.

- $*$ is *commutative* if $a * b = b * a$ for all $a, b \in A$.
- $*$ is *associative* if $(a * b) * c = a * (b * c)$.

Example

1. Multiplication and addition give operators on $\mathbb{Z}$ which are both commutative and associative.
2. Subtraction is an operation on $\mathbb{Z}$ which is neither commutative nor associative.
3. The binary operation on $\mathbb{Z}$ given by $x * y = 1 + xy$ is commutative but not associative. For example $(1 * 2) * 3 = 3 * 3 = 10$ while $1 * (2 * 3) = 1 * (7) = 8$.

Definition

Suppose that $*$ is a binary operation on a nonempty set A and that $B \subseteq A$. If it is true that $a * b \in B$ for all $a, b \in B$, then we say that B is closed under $*$.

Example

Consider addition on $\mathbb{Z}$. The set of even integers is closed under addition.

Proof.

Suppose that $a, b \in \mathbb{Z}$ are even.
Then there are $x, y \in \mathbb{Z}$ such that $a = 2x$ and $b = 2y$.
Thus $a + b = 2x + 2y = 2(x + y)$ which is even.
Since a and b were arbitrary even integers, it follows that the set of even integers is closed under addition. $\square$

Definition

Let $*$ be a binary operation on a nonempty set A. An element e is called an identity element with respect to $*$ if

$$e * x = x = x * e$$

for all $x \in A$.

Example

1. 1 is an identity element for multiplication on the integers.
2. 0 is an identity element for addition on the integers.
3. If $*$ is defined on $\mathbb{Z}$ by $x * y = x + y + 1$ Then $\underline{-1}$ is the identity.
4. The operation $*$ defined on $\mathbb{Z}$ by $x * y = 1 + xy$ has no identity element.

Definition

Suppose that $*$ is a binary operation on a nonempty set A and that e is an identity element with respect to $*$. Suppose that $a \in A$.

- If there exists $b \in A$ such that $a * b = e$ then b is called a *right inverse* of a with respect to $*$.
- If there exists $b \in A$ such that $b * a = e$ then b is called a *left inverse* of a with respect to $*$.
- If $b \in A$ is both a right and left inverse of a with respect to $*$ then we simply say that b is an *inverse* of a and we say that a is *invertible*.

Example

1. Consider the operation of addition on the integers. For any integer a, the inverse of a with respect to addition is $-a$.
2. Consider the operation of multiplication on $\mathbb{Z}$. The invertible elements are 1 and -1 .

Fact

Suppose that $*$ is a binary operation on a nonempty set A. If there is an identity element with respect to $*$ then it is unique. In the case that there is an identity element and that $*$ is associative then for each $a \in A$ if there is an inverse of a then it is unique.