MTHSC 3190 Section 1.3

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Definition

A <u>theorem</u> is a declarative statement which is known to always be true without exception and which has a proof (mathematically acceptable explanation).

Examples

Non-Theorem It is raining.

Non-Theorem In July, Clemson is hot.

Non-Theorem When an object is dropped near the earth, it accelerates at a rate of 9.8 m/s^2.

Theorem (*Pythagoras*) If a and b are the lengths os the legs of a right triangle and if c is the length of the hypotenuse, then

$$a^2 + b^2 = c^2.$$

Note

Many theorems in mathematics can be expressed in the form:

If A **then** B.

where A and B are themselves declarative statements

Example

Express the fact that the sum of two even integers is again even as an *if-then* statement.

Truth of If-Then statements

Consider the truth of the following statements as *English* statements and as *Mathematics* statements

1. If you don't finish your dinner then you will get no dessert.
2. If you clean your room, then I will give you $ 10.

Note

When speaking English, we sometimes hear or even intend two cause and effect statements in an *if-then* statement. However, in Mathematics, an *if-then* statement only expresses one such statement.

Truth of If-Then statements

The truth value in all instances of the statement If A then B is given by the following table.

A	B	**If** A **then** B
false	*false*	*true*
false	*true*	*true*
true	*false*	*false*
true	*true*	*true*

Notation

The following are notation for expressing *If A then B*.

1. A implies B
2. B is implied by A
3. B, if A
4. A is sufficient for B.
5. A is a sufficient condition for B
6. In order for B to hold, it is enough that we have A.
7. B is necessary for A. In order for A to be true, B must also be true (but it is possible that B is true and A is not true).
8. A, only if B.
9. $A \Rightarrow B$. The symbol $\Rightarrow$ is read as implies.
10. $B \Leftarrow A$. The symbol $\Leftarrow$ is read as implied by.

If-and-only-if statements

In mathematics the statement $\underline{A \text{ if and only if } B}$ means both of the following

1. If A then B.
2. If B then A.

Example

What two things are meant by "*An integer x is even if and only if the integer $x+1$ is odd.*" ?

The truth of *If-and-only-if statements*

The truth value in all instances of the statement *A if and only if B* is given by the following table

A	B	A **if and only if** B
false	*false*	*true*
false	*true*	*false*
true	*false*	*false*
true	*true*	*true*

Note

Note that the statement A if and only if B is true precisely when A and B have the same truth value. Thus we say that A if and only if B is true when A and B are logically equivalent.

And, Or and Not

The mathematical use of the words And, Or and Not is given by the following tables.

A	B	A **and** B
false	*false*	*false*
false	*true*	*false*
true	*false*	*false*
true	*true*	*true*

A	B	A **or** B
false	*false*	*false*
false	*true*	*true*
true	*false*	*true*
true	*true*	*true*

A	**Not** A
false	true
true	*false*

True statements

Other words used to denote true declarative statements are: *Result*, *Fact*, *Proposition*, *Lemma*, *Corollary*, *Claim*.

Vacuous Truth

Statements of the form If A then B for which A can never be true are said to be vacuously true.

Example

The statement
If x is both a perfect square and a prime, then it is negative is vacuously true.