MTHSC 3190 Section 2.9 Sets –a first look

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Definition

A set is a repetition free unordered collection of objects called elements.

Notation

If an object x is in a set S then we write $x \in S$ (read x is in S). Otherwise, we write $x \notin S$.

Example

1. $\{1, 2, 3\}, \{1, 5, 7\}, \{A, B, 2, \gamma, \mathbb{Z}\}$ are sets.
2. Let $S = \{1, 2, 3\}$. Then $1 \in S$, $2 \in S$ but $5 \notin S$.
3. $\emptyset = \{\}$ is the empty set.

Note

Given and object x and a set S, either $x \in S$ or $x \notin S$. There is no concept of x being in S multiple times. For example, $S = \{1, 2, 1\}$ is a set of size 2.

Definition

If A is a set, then the number $|A|$ of elements in A is the cardinality of A. This is also denoted $\#A$.

Set Notation

We typically define sets using set notation which looks like:

$$S = \left\{ \begin{array}{l} \\ \text{variable including type of} \ : \ \text{condition list} \\ \text{allowable objects (-e.g.} \\ \text{integers)} \end{array} \right\}$$

This notation means all instances of the variables which satisfy the condition list are in the set S. That is, the condition list is a membership test.

Example

1. $\mathcal{E} = \{x \in \mathbb{Z} \ : \ 2|x\}$ denotes the set of even integers.
2. $\mathcal{NE} = \{x \in \mathbb{Z} \ : \ 2 \nmid x\}$ denotes the set of integers which are not even. (One might expect that this is the same as the set of odd integers, but we have not yet proved this).
3. $\mathcal{O} = \{x \in \mathbb{Z} \ : \ x = 2k+1 \quad \text{for some } k \in \mathbb{Z}\}$ denotes the set of odd integers.

Set Equality

Two sets S and T are equal provided that for all (reasonable objects) x, $x \in S \Leftrightarrow x \in T$.

Subset

Given two sets S and T, we say that S is a <u>subset</u> of T provided that $x \in S \Rightarrow x \in T$. In this case, we may write $S \subseteq T$.

Example

Suppose that

$$\begin{aligned} S &= \{1, 3, 5, 7\} \qquad T = \{2, 4, 6, 8\} \\ R &= \{1, 2, 3, 4, 5, 6, 7, 8\} \qquad W = \{z \in \mathbb{Z} \ : \ 1 \leq z \leq 8\} \end{aligned}$$

Which of these sets are equal or subsets?

Note

If $S = \{x, y, z\}$ then we can write $x \in S$ or $\{x\} \subset S$.

Proof Template for Subset

Given two sets A and B we show $A \subseteq B$ as follows.

1. First choose an arbitrary (undetermined) element $x \in A$. We indicate this with a statement like "Let $x \in A$".
2. We conclude the proof by deducing that $x \in B$ and note that this implies that $A \subseteq B$. We indicate this with statements like "Therefore, $x \in B$. Thus $A \subseteq B$.
3. Expand from the definition of A what it means to be in A.
4. Expand from the bottom by explaining what it means to be in B.
5. Connect the two ends.

Example

Let $S = \{x \in \mathbb{Z} \ : \ 5|x\}$ and $T = \{x \in \mathbb{Z} \ : \ 50|x\}$. Show that $T \subseteq S$.

Proposition

Let A be a set. Then $x \in A \Leftrightarrow \{x\} \subseteq A$.

Proof.

Exercise $\square$

Proof Template for Set Equality

Let A and B be sets. To show that $A = B$ we must show that

1. $A \subseteq B$,
2. $B \subseteq A$.

Proposition

Let $E = \{x \in \mathbb{Z} \ : \ x \text{ is even}\}$ *and* $F = \{z \in \mathbb{Z} \ : \ z = a + b \text{ where } a, b \text{ are odd}\}$. *Then* $E = F$.

Proof.

Exercise □

Definition (Pythagorean Triple)

A list of 3 integers (a, b, c) is called a Pythagorean triple provided that $a^2 + b^2 = c^2$.

Example

$(3, 4, 5)$ and $(5, 12, 13)$ are Pythagorean triples

Proposition

Let P denote the set of all Pythagorean triples (-i.e. $P = \{(a, b, c) \ : \ a, b, c \in \mathbb{Z}; a^2 + b^2 = c^2\}$). Let $T = \{(p, q, r) \ : \ p, q, r \in \mathbb{Z}; p = x^2 - y^2, \ q = 2xy$ and $r = x^2 + y^2$ where $x, y, z \in \mathbb{Z}\}$. (We could also write $T = \{(x^2 - y^2, 2xy, x^2 + y^2) \ : \ x, y, z \in \mathbb{Z}\}$. Then $T \subseteq P$.

Proof.

Exercise. $\square$

Example

Let $A = \{1, 2, 3\}$. List the subsets of A.

Size of subsets	Subsets	Number of subsets of this size
0	$\emptyset$	1
1	$\{1\}$, $\{2\}$, $\{3\}$	3
2	$\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$	3
3	$\{1, 2, 3\}$	1
	Total $=$	8

Theorem

Let A be a finite set. The number of subsets of A is $2^{|A|}$.

Definition (Power Set)

Let A be a set. The power set of A is denoted 2^A and is defined to be the set of all subsets of A.

Example

Let $A = \{1, 2\}$. Then

$$2^A = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}.$$

Note

In our example $\{1\} \in 2^A$ and $\{\{1\}\} \subseteq 2^A$.

Corollary

If A is a finite set, then $|2^A| = 2^{|A|}$.