MTHSC 3190 Section 10 Quantifiers

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Definition (Existential Quantifiers and Statements)

An existential statement asserts the existence of an object of some type with certain specified properties. Such a statement typically is preceded with an existential quantifier such as "there exists" or "there is".

Example

There is an integer x that is divisible by 2.

General Form

There is $x \in A$ such that P.

Notation

The existential quantifiers there is or there exists are sometimes replaced with the notation $\exists$.

Proof Template for Existence

To prove the statement "$\underline{\exists x \in A \text{ such that } P}$", we simply need to find an example of an element of A which satisfies P. Our proof will be constructed as follows.

1. Select a value for the variable x by stating "Let $x =$ ______."
2. Verify that x satisfies the claimed properties P.
3. Close the proof with "Therefore $x =$ ______ satisfies P as claimed".

Example

Show that there is $x \in \mathbb{Z}$ such that $2|x$.

Definition (Universal Quantifiers and Statements)

A universal statement asserts that a certain property is satisfied by all members of a certain set (or universe). Universal statements are typically preceded by a universal quantifier such as "for all", "every", "all", "each", ...

Example

1. Every integer is either even or odd.
2. All integers are . . . .
3. Each integer is . . . .
4. Let $x \in \mathbb{Z}$, then . . . .

Notation

The symbol $\forall$ read "for all" is often used as to denote the universal quantifier

General Form

The general form of a statement with a universal quantifier is

$$\forall x \in A,\ S(x),$$

where $S(x)$ is a statement about x.

Example

$\forall x \in \mathbb{Z}$, x is even or x is odd.

Proof template for Universal statements

To prove the statement "$\forall x \in A,\ S(x)$" we proceed as follows

1. Start the proof by establishing notation for an arbitrary element of A with a statement like "Let $x \in A$" or "Suppose $x \in A$."
2. Prove the statement $S(x)$ about x using an appropriate proof template.

Negation of universal and existential statements

- $\neg(\exists x \in A$ such that $P(x)) = \forall x \in A, \neg P(x)$.
- $\neg(\forall x \in A, S(x)) = \exists x \in A$ such that $\neg S(x)$.

Example

1. Express the statement "There is no integer that is both even and odd." using quantifiers.
2. Express the statement "Not all integers are prime." using quantifiers.

Consider the following statements. Are they true?

1. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $xy = 1$.
2. $\exists y \in \mathbb{R}$ such that $\forall x \in \mathbb{R}, xy = 1$.