MTHS 3190
Exam #1
January 29, 2015

Name: *Examples*

Sequence Example: Compute $\lim_{n\to\infty} \frac{n^3+2n^2+1}{2n^3}$.

Solution: First, we must decide what the limit is. Let's rewrite the expression

$$\frac{n^3+2n^2+1}{2n^3} = \frac{1}{2} + \frac{1}{n} + \frac{1}{2n^3}.$$

Since all of the terms except the first one tend to zero as n gets large, we will guess that the limit is $\frac{1}{2}$. Now, we must prove it.

Scratch Work: We will need to consider the difference of the n^{th} term of our sequence and the claimed limit which is

$$\frac{n^3+2n^2+1}{2n^3} - \frac{1}{2} = \frac{1}{n} + \frac{1}{2n^3}$$

and show that it is small. That is we need to solve

$$\frac{1}{n} + \frac{1}{2n^3} < \epsilon$$

for n.

We will use a fairly common trick to do this. We will replace our expression with a less complicated but slightly larger one and prove this new expression is smaller than ϵ.

Note: For $n \geq 1$, we have

$$\begin{aligned} 1 &\leq n \\ \Rightarrow \frac{1}{n} &\leq 1 \\ \Rightarrow \frac{1}{n^2} &\leq \frac{1}{n} \\ \Rightarrow \frac{1}{n^3} &\leq \frac{1}{n^2}. \end{aligned}$$

Thus, we have $\frac{1}{n^3} \leq \frac{1}{n^2} \leq \frac{1}{n}$ for all $n \geq 1$. Since we are interested in large values of n we may use these inequalities as follows.

$$\frac{n^3+2n^2+1}{2n^3} - \frac{1}{2} = \frac{1}{n} + \frac{1}{2n^3} = \frac{1}{n} + \frac{1}{2}\cdot\frac{1}{n^3} \leq \frac{1}{n} + \frac{1}{2}\cdot\frac{1}{n} = \frac{3}{2n}.$$

Now, $\frac{3}{2n} < \epsilon \Leftrightarrow \frac{3}{2\epsilon} < n$.

So, we will take $N = \left\lceil \frac{3}{2\epsilon} \right\rceil + 1$.

Now, we are ready to reorganize this into a proof (basically by writing our scratch work backwards). We will of course be sure to destroy our scratch work afterwards to make ourselves appear much smarter.

Proof. Let $\epsilon > 0$.
Take $N = \left\lceil \frac{3}{2\epsilon} \right\rceil + 1$.
Let $n \geq N$.
Then we first note that $n \geq N \geq 2$. Thus, we have

$$\begin{aligned} 1 &< n \\ \Rightarrow \frac{1}{n} &< 1 \\ \Rightarrow \frac{1}{n^2} &< \frac{1}{n} \\ \Rightarrow \frac{1}{n^3} &< \frac{1}{n^2}. \end{aligned}$$

Also, $n \geq N > \frac{3}{2\epsilon}$ which implies that $\epsilon > \frac{3}{2n}$.
Thus, we have

$$\left| \frac{n^3 + 2n^2 + 1}{2n^3} - \frac{1}{2} \right| = \left| \frac{1}{n} + \frac{1}{2n^3} \right| \leq \frac{1}{n} + \frac{1}{2n} = \frac{3}{2n} < \epsilon.$$

Since ϵ and $n \geq N$ were arbitrary, we have shown that

$$\lim_{n \to \infty} \frac{n^3 + 2n^2 + 1}{2n^3} = \frac{1}{2}.$$

$\square$

Series Example: Compute $\sum_{n=0}^{\infty} \left(\frac{11}{13}\right)^n$ if it exists and prove your claims.

Solution: First we must determine if the series converges and to what if it does. We will use our formula from class, namely

$$\sum_{n=0}^{k} x^n = \frac{x^{k+1}-1}{x-1},$$

which is valid for $x \neq 0, 1$.

Taking $x = \frac{11}{13}$, we have

$$\sum_{n=0}^{k} \left(\frac{11}{13}\right)^n = \frac{\left(\frac{11}{13}\right)^{k+1}-1}{\frac{11}{13}-1} = \frac{13}{2}\left(1-\left(\frac{11}{13}\right)^{k+1}\right) = \frac{13}{2} - \frac{11}{2}\cdot\left(\frac{11}{13}\right)^k.$$

Now we note that as k gets large this last expression gets closer to $\frac{13}{2}$. So, we guess that

$$\sum_{n=0}^{\infty} \left(\frac{11}{13}\right)^n = \lim_{k\to\infty}\left[\sum_{n=0}^{k}\left(\frac{11}{13}\right)^n\right] = \lim_{k\to\infty}\left[\frac{13}{2} - \frac{11}{2}\cdot\left(\frac{11}{13}\right)^k\right] = \frac{13}{2}.$$

Now let's prove it.

Scratch Work First, we do a little scratch work to figure out which N to choose in our argument. We need to again consider the difference in the k^{th} sequence term (or partial sum) and our claimed limit and argue that it is small. That is we will consider

$$\begin{aligned}
\left|\sum_{n=0}^{k}\left(\frac{11}{13}\right)^n - \frac{13}{2}\right| &= \left|\frac{\left(\frac{11}{13}\right)^{k+1}-1}{\frac{11}{13}-1} - \frac{13}{2}\right| \\
&= \left|\frac{13}{2}\left(1-\left(\frac{11}{13}\right)^{k+1}\right) - \frac{13}{2}\right| \\
&= \frac{11}{2}\cdot\left(\frac{11}{13}\right)^k
\end{aligned}$$

Thus

$$\begin{aligned}
&\left|\sum_{n=0}^{k}\left(\frac{11}{13}\right)^n - \frac{13}{2}\right| < \epsilon \\
\Leftrightarrow\ & \frac{11}{2}\cdot\left(\frac{11}{13}\right)^k < \epsilon \\
\Leftrightarrow\ & \left(\frac{11}{13}\right)^k < \frac{2\epsilon}{11} \\
\Leftrightarrow\ & k\log\left(\frac{11}{13}\right) < \log\left(\frac{2\epsilon}{11}\right) \\
\Leftrightarrow\ & k > \frac{\log\left(\frac{2\epsilon}{11}\right)}{\log\left(\frac{11}{13}\right)}. \qquad \text{(Note that } \log\left(\tfrac{11}{13}\right) < 0\text{. So, our inequality changes.)}
\end{aligned}$$

So we will take $N = \text{Max}\left(1, \left\lceil \frac{\log\left(\frac{2\epsilon}{11}\right)}{\log\left(\frac{11}{13}\right)} \right\rceil + 1\right)$.

Now, we are ready to write up our proof.

Proof. Let $\epsilon > 0$. Take $N = \text{Max}\left(1, \left\lceil \frac{\log\left(\frac{2\epsilon}{11}\right)}{\log\left(\frac{11}{13}\right)} \right\rceil + 1\right)$.
Let $n \geq N$.
Then $n \geq N > \frac{\log\left(\frac{2\epsilon}{11}\right)}{\log\left(\frac{11}{13}\right)}$.
Thus, $\log\left(\left(\frac{11}{13}\right)^n\right) < \log\left(\frac{2\epsilon}{11}\right)$,
which implies that $\left(\frac{11}{13}\right)^n < \frac{2\epsilon}{11}$.
Thus, $\frac{11}{2} \cdot \left(\frac{11}{13}\right)^n < \epsilon$.
So, for $n \geq N$, we have that

$$\begin{aligned}
\left|\sum_{m=0}^{n} \left(\frac{11}{13}\right)^m - \frac{13}{2}\right| &= \left|\frac{\left(\frac{11}{13}\right)^{n+1} - 1}{\frac{11}{13} - 1} - \frac{13}{2}\right| \\
&= \left|\frac{13}{2}\left(1 - \left(\frac{11}{13}\right)^{n+1}\right) - \frac{13}{2}\right| \\
&= \frac{11}{2} \cdot \left(\frac{11}{13}\right)^n < \epsilon.
\end{aligned}$$

Thus we have shown that

$$\sum_{m=0}^{\infty} \left(\frac{11}{13}\right)^m = \lim_{n\to\infty}\left[\sum_{m=0}^{n} \left(\frac{11}{13}\right)^m\right] = \frac{13}{2}$$

$\square$