MTHSC 3190 Section 1.10 - Analysis Supplement

Sequences, Series and Limits

Kevin James

Definition

We define a sequence to be an infinite list. We will usually denote sequences as

$$(s_n)_{n\geq 1} = (s_1, s_2, s_3, \ldots).$$

Note

We will confine out attention to sequences of real numbers. Although one can certainly consider much of our discussion in other settings such as the complex numbers.

Definition

We say that a sequence s_n of real numbers converges to a limit $L \in \mathbb{R}$ provided that

$$\forall \epsilon > 0, \exists N \in \mathbb{Z}, \forall n \geq N, \; |s_n - L| < \epsilon.$$

Example

Let $s_n = \frac{1}{n}$.
Then, $\lim_{n \to \infty} s_n = 0$.

Proof

Let $\epsilon > 0$.
Take $N = \left\lceil \frac{1}{\epsilon} \right\rceil + 1$.
Suppose that $n \geq N$.
Then $n > \frac{1}{\epsilon}$.
Thus $|s_n - 0| = \frac{1}{n} < \epsilon$ □

Exercise

Let $s_n = \frac{3n^2+2n+1}{n^2}$. What is the limit? Prove it.

Definition

For a sequence $(a_n)_{n\geq 1}$ of real numbers we associate the series

$$\sum_{n\geq 1} a_n$$

We also have the associated sequence of partial sums given by

$$S_k = \sum_{n=1}^{k} a_n$$

Definition

We say that the series $\sum_{n\geq 1} a_n$ converges to a limit $L \in \mathbb{R}$ provided that its sequence of partial sums $(S_n)_{n\geq 1}$ converges to L.

Example

Consider the series $\sum_{n\geq 0} \frac{1}{2^n}$.
You may use the fact that for any $0 < x \neq 1$,

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1}$$

which we will prove later by induction.
Show that the above series series converges to 2.

Exercise

1. Compute $\sum_{n\geq 0}\left(\frac{2}{3}\right)^n$.
2. Compute $\sum_{n\geq 0}\left(\frac{1}{9}\right)^n$.
3. Compute $\sum_{n\geq 2}\left(\frac{1}{9}\right)^n$.
4. Show that $\sum_{n\geq 0}(-1)^n$ does not exist. Note that this will involve negating the statement that the limit does exist which involves 3 quantifiers. Be careful.

Definition

1. We say that a sequence $(s_n)_{n\geq 1}$ increases without bound or that it has limit ∞ and write
$$\lim_{n\to\infty}[s_n] = \infty,$$
provided that
$$\forall B > 0, \exists N \in \mathbb{N}, \forall n \geq N, s_n > B.$$
2. We say that a sequence $(s_n)_{n\geq 1}$ decreases without bound or that it has limit $-\infty$ and write
$$\lim_{n\to\infty}[s_n] = -\infty,$$
provided that
$$\forall B < 0, \exists N \in \mathbb{N}, \forall n \geq N, s_n > B.$$

Example

Compute the following limits and prove your answer.

1. $\lim_{n\to\infty} n = \infty$.
2. $\lim_{n\to\infty} \left(\frac{1-2n^2}{n+1}\right) = -\infty$.
3. $\sum_{n\geq 1} 1 = \infty$.
4. $\sum_{n\geq 1} \frac{1}{n} = \infty$.

Definition

Given a sequence $(s_n)_{n\geq 1}$ we say that the sequence has a limit or that the limit of the sequence exists provided that

$$\exists L, \forall \epsilon > 0, \exists N \in \mathbb{Z}, \forall n \geq N, \ |s_n - L| < \epsilon.$$

Definition

Given a sequence $(s_n)_{n\geq 1}$ we say that the sequence has no limit or that the limit of the sequence does not exist provided that

$$\forall L, \exists \epsilon > 0, \forall N \in \mathbb{Z}, \exists n \geq N, \ |s_n - L| \geq \epsilon.$$

Example

Show that the following sequences have no limit.

1. $s_n = (-1)^n$.
2. $s_n = \sin\left(\frac{2\pi n}{100}\right)$.

Definition

We call a sequence $(s_n)_{n\geq 1}$ of real numbers a Cauchy sequence if

$$\forall \epsilon > 0, \exists N \in \mathbb{Z}, \forall m, n > N, |s_m - s_n| < \epsilon.$$

Exercise

1. Show that if a sequence converges then it is Cauchy.
2. Which of the sequences that we have considered are Cauchy? Prove your assertions.

Definition

We say that a series $\sum_{n\geq 1} a_n$ is a Cauchy Series if its sequence of partial sums is a Cauchy sequence. Alternatively, this means that

$$\forall \epsilon > 0, \exists N \in \mathbb{Z}, \forall n > m > N, \left| \sum_{k=m+1}^{n} a_k \right| < \epsilon$$

Exercise

Which of the series considered before are Cauchy? Prove your assertions.