MTHSC 3190 Section 2.11

Kevin James

DEFINITION

Suppose that A and B are sets. Then we define the <u>union</u> denoted by $A \cup B$ and <u>intersection</u> denoted by $A \cap B$ as

$$A \cup B = \{x : x \in A \text{ or } x \in B\},\$$

 $A \cap B = \{x : x \in A \text{ and } x \in B\}.$

EXAMPLE

THEOREM

Let A, B and C be sets. Then,

Proof

Make sure you can prove all of these. We will prove a couple now...

VENN DIAGRAMS

Note

Venn diagrams are very useful in visualizing set operations and often lead us to discover true statements about sets and their operations. However a sequence of Venn diagrams WILL NOT be accepted as a proof.

EXAMPLE

Draw the Venn diagrams which represent $A, B, A \cup B$ and $A \cap B$.

Inclusion/Exclusion - first case

Proposition

Suppose that A and B are sets. Then,

$$|A| + |B| = |A \cup B| + |A \cap B|.$$

COROLLARY (INCLUSION/EXCULUSION)

$$|A \cup B| = |A| + |B| - |A \cap B|$$
.

COMBINATORIAL PROOF

Proof

Let us label the elements of $A \cup B$ in the following 2 ways.

First, we will label each element of A with an A.

We then label each element of B with a \mathcal{B} .

How many labels were used?

Answer 1

Answer 2

Thus
$$A|+|B| = \text{Number of Labels} = |A \cup B| + |A \cap B|$$
.

Combinatorial Proof

PROOF TEMPLATE

A combinatorial proof is typically used to prove an identity of the form

$$LHS = RHS$$
.

We proceed as follows:

- Pose a counting question.
- 2 Argue that the LHS answers the question.
- 3 Argue that the RHS answers the question.
- 4 Conclude that

$$LHS = Answer = RHS$$
.

Set Operations

Union and Intersection
Counting
Difference and Symmetric Difference
Sets and Logic
Cartesian Product

Note

Finding the right question can be difficult. To get started ask yourself what is being counted by the LHS or the RHS.

Inclusion/Exclusion Application

How many integers $1 \le x \le 1000$ are divisible by 2 or 5?

DEFINITION

- **1** Let A and B be sets. We say that A and B are disjoint provided that $A \cap B = \emptyset$.
- **2** Let A_1, \ldots, A_n be a collection of sets. This collection is said to be pairwise disjoint provided that whenever $i \neq j$, $A_i \cap A_j = \emptyset$.

EXAMPLE

Let $A = \{1, 2, 3\}, B = \{4, 5, 6\}$, and $C = \{7, 8, 9\}$. Check that this collection of 3 sets is pairwise disjoint.

COROLLARY

Let A and B be disjoint sets. Then $|A \cup B| = |A| + |B|$.

COROLLARY

Suppose that A_1, \ldots, A_n is a pairwise disjoint collection of sets. Then,

$$|\cup_{i=1}^n A_i| = \sum_{i=1}^n |A_i|.$$

DEFINITION

Let A and B be sets.

- **1** We define their <u>difference</u> $A \setminus B$ as $A \setminus B = \{x \in A : x \notin B\}$.
- **2** We define their symmetric difference $A \triangle B$ as $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Note

The notation A - B is sometimes used in place of $A \setminus B$.

EXAMPLE

Let $A = \{1, 2, 3, 4\}$ and let $B = \{1, 4, 7, 9\}$.

Compute $A \setminus B$, $B \setminus A$, $A \triangle B$ and $B \triangle A$.

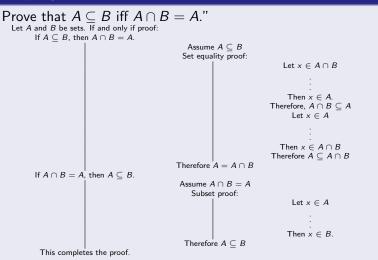
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EXAMPLE

Draw the Venn diagram for $A \triangle B$. Is there another convenient description/formula for this operator? Prove your formula.

ADVANCED EXAMPLE



Proposition

Let A, B, and C be sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof.

Let A, B, and C be sets such that $A \subseteq B$ and $B \subseteq C$. We will prove, by element chasing, that $A \subseteq C$. Suppose that $x \in A$. Since $x \in A$ and $A \subseteq B$, $x \in B$. Since $x \in B$ and $B \subseteq C$, $x \in C$. Therefore, $x \in C$ and $A \subseteq C$.

EXAMPLE

Now prove it using a truth table.

DEFINITION

Let A and B be sets. The Cartesian product $A \times B$ is defined as

$$A\times B=\{(a,b)\ :\ a\in A;b\in B\}.$$

EXAMPLE

Take $A = \{1, 2, 3\}$ and $B = \{x, y\}$. What is $A \times B$?

Proposition

Let A and B be finite sets. Then $|A \times B| = |A||B|$.