MTHSC 3190 Section 2.11

Kevin James

Definition

Suppose that A and B are sets. Then we define the union denoted by $A \cup B$ and intersection denoted by $A \cap B$ as

$$\begin{aligned} A \cup B &= \{x : x \in A \text{ or } x \in B\}, \\ A \cap B &= \{x : x \in A \text{ and } x \in B\}. \end{aligned}$$

Example

$A = \{a, b, c, d\}$, $B = \{c, d, e, f\}$
$A \cup B =$ ________
$A \cap B =$ ________

Theorem

Let A, B and C be sets. Then,

1. $A \cup B = B \cup A$; $A \cap B = B \cap A$.
2. $(A \cup B) \cup C = A \cup (B \cup C)$; $(A \cap B) \cap C = A \cap (B \cap C)$.
3. $A \cup \emptyset = A$; $A \cap \emptyset = \emptyset$.
4. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
5. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof

Make sure you can prove all of these. We will prove a couple now...

Venn Diagrams

Note

Venn diagrams are very useful in visualizing set operations and often lead us to discover true statements about sets and their operations. However a sequence of Venn diagrams *WILL NOT be accepted as a proof.*

Example

Draw the Venn diagrams which represent $A, B, A \cup B$ and $A \cap B$.

Inclusion/Exclusion – first case

Proposition

Suppose that A and B are sets. Then,

$$|A| + |B| = |A \cup B| + |A \cap B|.$$

Corollary (Inclusion/Exculusion)

$|A \cup B| = |A| + |B| - |A \cap B|.$

Combinatorial Proof

Proof

Let us label the elements of $A \cup B$ in the following 2 ways.
First, we will label each element of A with an $\mathcal{A}$.
We then label each element of B with a $\mathcal{B}$.
How many labels were used?

Answer 1

Answer 2

Thus $A| + |B| = \text{Number of Labels} = |A \cup B| + |A \cap B|$.

Combinatorial Proof

Proof Template

A combinatorial proof is typically used to prove an identity of the form

$$LHS = RHS.$$

We proceed as follows:

1. Pose a counting question.
2. Argue that the LHS answers the question.
3. Argue that the RHS answers the question.
4. Conclude that
$$LHS = \text{Answer} = RHS.$$

Note

Finding the right question can be difficult. To get started ask yourself what is being counted by the LHS or the RHS.

Inclusion/Exclusion Application

How many integers $1 \leq x \leq 1000$ are divisible by 2 or 5?

Definition

1. Let A and B be sets. We say that A and B are disjoint provided that $A \cap B = \emptyset$.
2. Let $A_1, \ldots, A_n$ be a collection of sets. This collection is said to be pairwise disjoint provided that whenever $i \neq j$, $A_i \cap A_j = \emptyset$.

Example

Let $A = \{1, 2, 3\}, B = \{4, 5, 6\}$, and $C = \{7, 8, 9\}$. Check that this collection of 3 sets is pairwise disjoint.

Corollary

Let A and B be disjoint sets. Then $|A \cup B| = |A| + |B|$.

Corollary

Suppose that $A_1, \ldots, A_n$ is a pairwise disjoint collection of sets. Then,

$$|\cup_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i|.$$

Definition

Let A and B be sets.

1. We define their difference $A \setminus B$ as $A \setminus B = \{x \in A \ : \ x \notin B\}$.
2. We define their symmetric difference $A \triangle B$ as $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Note

The notation $A - B$ is sometimes used in place of $A \setminus B$.

Example

Let $A = \{1, 2, 3, 4\}$ and let $B = \{1, 4, 7, 9\}$.
Compute $A \setminus B$, $B \setminus A$, $A \triangle B$ and $B \triangle A$.

Example

Draw the Venn diagram for $A \triangle B$. Is there another convenient description/formula for this operator? Prove your formula.

Advanced Example

Prove that $A \subseteq B$ iff $A \cap B = A$."

Let A and B be sets. If and only if proof:

If $A \subseteq B$, then $A \cap B = A$.

Assume $A \subseteq B$
Set equality proof:

Let $x \in A \cap B$

$\vdots$

Then $x \in A$.
Therefore, $A \cap B \subseteq A$
Let $x \in A$

$\vdots$

Then $x \in A \cap B$
Therefore $A \subseteq A \cap B$

Therefore $A = A \cap B$

If $A \cap B = A$, then $A \subseteq B$.

Assume $A \cap B = A$
Subset proof:

Let $x \in A$

$\vdots$

Then $x \in B$.

Therefore $A \subseteq B$

This completes the proof.

Proposition

Let A, B, and C be sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof.

Let A, B, and C be sets such that $A \subseteq B$ and $B \subseteq C$. We will prove, by element chasing, that $A \subseteq C$. Suppose that $x \in A$. Since $x \in A$ and $A \subseteq B$, $x \in B$. Since $x \in B$ and $B \subseteq C$, $x \in C$. Therefore, $x \in C$ and $A \subseteq C$. □

Example

Now prove it using a truth table.

Definition

Let A and B be sets. The Cartesian product $A \times B$ is defined as

$$A \times B = \{(a, b) \ : \ a \in A; b \in B\}.$$

Example

Take $A = \{1, 2, 3\}$ and $B = \{x, y\}$. What is $A \times B$?

Proposition

Let A and B be finite sets. Then $|A \times B| = |A||B|$.