

# MTHSC 3190 SECTION 2.11

Kevin James

## DEFINITION

Suppose that  $A$  and  $B$  are sets. Then we define the union denoted by  $A \cup B$  and intersection denoted by  $A \cap B$  as

$$A \cup B = \{x : x \in A \text{ or } x \in B\},$$

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## EXAMPLE

$$A = \{a, b, c, d\}, B = \{c, d, e, f\}$$

$$A \cup B = \underline{\hspace{2cm}}$$

$$A \cap B = \underline{\hspace{2cm}}$$

## THEOREM

Let  $A, B$  and  $C$  be sets. Then,

- 1  $A \cup B = B \cup A; A \cap B = B \cap A.$
- 2  $(A \cup B) \cup C = A \cup (B \cup C); (A \cap B) \cap C = A \cap (B \cap C).$
- 3  $A \cup \emptyset = A; A \cap \emptyset = \emptyset.$
- 4  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$
- 5  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$

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## PROOF

Make sure you can prove all of these. We will prove a couple now...

## VENN DIAGRAMS

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Venn diagrams are very useful in visualizing set operations and often lead us to discover true statements about sets and their operations. However a sequence of Venn diagrams *WILL NOT* be accepted as a proof.

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## EXAMPLE

Draw the Venn diagrams which represent  $A$ ,  $B$ ,  $A \cup B$  and  $A \cap B$ .

## INCLUSION/EXCLUSION – FIRST CASE

## PROPOSITION

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## COROLLARY (INCLUSION/EXCLUSION)

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

## COMBINATORIAL PROOF

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Thus  $|A| + |B| = \text{Number of Labels} = |A \cup B| + |A \cap B|.$



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## PROOF TEMPLATE

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$$LHS = RHS.$$

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- 2 Argue that the LHS answers the question.
- 3 Argue that the RHS answers the question.
- 4 Conclude that

$$LHS = \text{Answer} = RHS.$$

## NOTE

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## INCLUSION/EXCLUSION APPLICATION

How many integers  $1 \leq x \leq 1000$  are divisible by 2 or 5?

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## EXAMPLE

Let  $A = \{1, 2, 3\}$ ,  $B = \{4, 5, 6\}$ , and  $C = \{7, 8, 9\}$ . Check that this collection of 3 sets is pairwise disjoint.

## COROLLARY

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Suppose that  $A_1, \dots, A_n$  is a pairwise disjoint collection of sets. Then,

$$|\cup_{i=1}^n A_i| = \sum_{i=1}^n |A_i|.$$

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Let  $A$  and  $B$  be sets.

- 1 We define their difference  $A \setminus B$  as  $A \setminus B = \{x \in A : x \notin B\}$ .
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## EXAMPLE

Let  $A = \{1, 2, 3, 4\}$  and let  $B = \{1, 4, 7, 9\}$ .

Compute  $A \setminus B$ ,  $B \setminus A$ ,  $A \Delta B$  and  $B \Delta A$ .

**EXAMPLE**

Draw the Venn diagram for  $A \Delta B$ . Is there another convenient description/formula for this operator? Prove your formula.

## ADVANCED EXAMPLE

Prove that  $A \subseteq B$  iff  $A \cap B = A$ ."

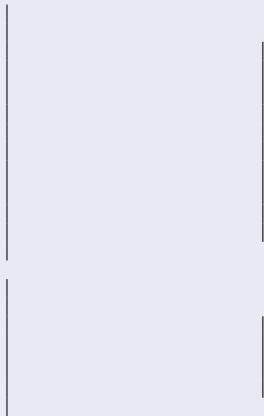




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Let  $x \in A \cap B$

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Then  $x \in A$ .

Therefore,  $A \cap B \subseteq A$

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## PROPOSITION

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## PROOF.

Let  $A$ ,  $B$ , and  $C$  be sets such that  $A \subseteq B$  and  $B \subseteq C$ . We will prove, by element chasing, that  $A \subseteq C$ . Suppose that  $x \in A$ . Since  $x \in A$  and  $A \subseteq B$ ,  $x \in B$ . Since  $x \in B$  and  $B \subseteq C$ ,  $x \in C$ . Therefore,  $x \in C$  and  $A \subseteq C$ . □

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## EXAMPLE

Now prove it using a truth table.

## DEFINITION

Let  $A$  and  $B$  be sets. The Cartesian product  $A \times B$  is defined as

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## PROPOSITION

*Let  $A$  and  $B$  be finite sets. Then  $|A \times B| = |A||B|$ .*