MTHSC 3190 Section 2.12

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Idea

A combinatorial proof is typically used to prove an identity of the form

$$LHS = RHS.$$

We proceed as follows:

1. Pose a counting question.
2. Argue that the LHS answers the question.
3. Argue that the RHS answers the question.
4. Conclude that

$$LHS = \text{Answer} = RHS.$$

Proposition

Let $0 < n \in \mathbb{Z}$. *Then*

$$\sum_{k=0}^{n-1} 2^k = 2^n - 1.$$

Note

1. The LHS counts all subsets of k-element sets k for $0 \le k \le n-1$.
2. The RHS counts the nonempty subsets of an n-element set.

Example

Let us consider the nonempty subsets of $\{1, 2, 3\}$ which have a given largest element.

Largest element	Subsets
1	
2	
3	

Proof

Let $n > 0$ and let $S = \{1, 2, \ldots, n\}$. How many nonempty subsets does S have?

Answer 1 There are 2^n subsets of S and only one of them is empty. Thus there are $2^n - 1$.

Answer 2 Consider the subsets whose largest element is j.
If T is such a subset, then $T \subset \{1, 2, \ldots, (j-1)\}$.
Thus there are 2^{j-1} such subsets.
Since each nonempty subset of S has a largest element, we have
$\#$ Nonemptysubsets $=$
$\sum_{j=1}^{n} \#$ subsets with largest element $j = \sum_{j=1}^{n} 2^{j-1}$.

Thus we conclude That

$$2^n - 1 = \#\text{ Nonempty subsets} = \sum_{j=1}^{n} 2^{j-1} = \sum_{k=0}^{n-1} 2^k \qquad \square$$