MTHSC 3190 Section 2.12

Kevin James

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4. Conclude that

$$LHS = \text{Answer} = RHS.$$

Proposition

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Note

1. The LHS counts all subsets of k-element sets k for $0 \leq k \leq n-1$.
2. The RHS counts the nonempty subsets of an n-element set.

Example

Let us consider the nonempty subsets of $\{1, 2, 3\}$ which have a given largest element.

Largest element	Subsets
1	
2	
3	

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