MTHSC 3190 Section 3.14

Kevin James

Definition

Let R be a relation on a set A. We say that R is an equivalence relation provided that R is

1. Reflexive
2. Symmetric
3. Transitive.

Example

Let $\mathcal{S}$ be the set of all finite sets. Let R be defined as follows. If A and B are sets, then $A\ R\ B \Leftrightarrow |A| = |B|$. Show that R is an equivalence relation on $\mathcal{S}$.

Proof Template

In order to show that a relation R on a set A is an equivalence relation we argue that it is reflexive, symmetric and transitive.

Proof.

Suppose that R is the relation on the set A defined by ....

Reflexive Let $x \in A$. .... Then, xRx.

Symmetric Suppose that $x, y \in A$ and xRy. .... Then yRx.

Transitive Suppose that $x, y, z \in A$, xRy and yRz. .... Then, xRz.

Since R is reflexive, symmetric and transitive it is an equivalence relation on A. □

Definition

Let $0 < n \in \mathbb{Z}$. We say that $x, y \in \mathbb{Z}$ are congruent modulo n and write

$$x \equiv y \pmod{n}$$

provided that $n | (x - y)$.

Example

- $3 \equiv 13 \pmod{5}$.
- $10 \equiv 200 \pmod{19}$.
- $-5 \equiv 3 \pmod{4}$.
- $16 \equiv 30 \pmod{7}$.

Theorem

Let $0 < n \in \mathbb{Z}$. The $\equiv \pmod{n}$ relation is an equivalence relation on $\mathbb{Z}$.

Proof.

Reflexive

Symmetric

Transitive

$\square$

Definition

Let R be an equivalence relation on a set A and let $a \in A$. We define the equivalence class of a, denoted by $[a]$ or $\bar{a}$ as

$$[a] = \{x \in A \ : \ xRa\}.$$

Example

Consider $\equiv \pmod 2$ on $\mathbb{Z}$.

$[1] =$

$[2] =$

$[3] =$

Example

Let $S = \{1, 2, 3\}$. Consider the "has the same size" relation on 2^S.

$[\emptyset] =$

$[\{2\}] =$

$[\{1, 3\}] =$

$[\{1, 2, 3\}] =$

Proposition

Let R be an equivalence relation on a set A and let $a \in A$. Then $a \in [a]$.

Proof.

$\square$

Corollary

Let R be an equivalence relation on a set A.

1. $[a] \neq \emptyset, \forall a \in A$.
2. $\bigcup_{a \in A}[a] = A$.

Proof.

$\square$

Proposition

Let R be an equivalence relation on a set A and let $a, b \in A$. Then $aRb \Leftrightarrow [a] = [b]$.

Proof.

$\square$

Proposition

Let R ve an equivalence relation on a set A and let $a, x, y \in A$. If $x, y \in [a]$, then xRy.

Proof.

Exercise $\square$

Proposition

Let R be an equivalence relation on a set A and suppose that $[a] \cap [b] \neq \emptyset$. Then $[a] = [b]$.

Proof.

$\square$

Corollary

Let R be an equivalence relation on a set A. The equivalence classes of R are nonempty, pairwise disjoint subsets of A whose union is A.