MTHSC 3190 Section 3.15

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Definition

Let A be a set. A partition $\mathcal{P}$ of A is a pairwise disjoint set of nonempty subsets of A whose union is A.

Example

$$
\begin{aligned}
A &= \{1, 2, 3, 4\} \\
\mathcal{P}_1 &= \{\{1, 3\}, \{2, 4\}\} \\
\mathcal{P}_2 &= \{\{1, 2, 3\}, \{4\}\}
\end{aligned}
$$

Observations

If $\mathcal{P}$ is a partition of A, then

1. the elements of $\mathcal{P}$ are subsets of A, that is $\mathcal{P} \subseteq 2^A$.
2. $\emptyset \notin \mathcal{P}$.
3. If $T, S \in \mathcal{P}$, then

$$T \cap S = \begin{cases} \emptyset, & \text{if } T \neq S, \\ T & \text{if } T = S. \end{cases}$$

4. $\bigcup_{S \in \mathcal{P}} S = A$.

Note

We can restate Cor 14.13 as follows. Let R be an equivalence relation on a set A. The equivalence classes of R form a partition of A.

Definition

Let $\mathcal{P}$ be a partition of a set A. Define an equivalence relation $\stackrel{\mathcal{P}}{\equiv}$ on A as follows:

$$\stackrel{\mathcal{P}}{\equiv} \; = \{(x, y) \; : \; \exists S \in \mathcal{P} \text{ such that } x, y \in S\},$$

that is $x \stackrel{\mathcal{P}}{\equiv} y$ if and only if x and y are in the same part of $\mathcal{P}$.

Proposition

Let A be a set and let $\mathcal{P}$ be a partition of A. The relation $\stackrel{\mathcal{P}}{\equiv}$ is an equivalence relation on A.

Proof.

Reflexive

Symmetric

Transitive

$\square$

Proposition

Let $\mathcal{P}$ be a partition of a set A and let $\overset{\mathcal{P}}{\equiv}$ be defined as before. The equivalence classes of $\overset{\mathcal{P}}{\equiv}$. are precisely the parts of $\mathcal{P}$.

Proof.

Exercise.
Hint: Let $\mathcal{E} = \{[a] \ : \ a \in A\}$.
Show that $\mathcal{E} = \mathcal{P}$ using the set equality template. $\square$

Example

Let $A = 2^{\{1,2,3\}}$ and let $\sim$ be the "has the same size" relation. Let $\mathcal{P}$ be the set of equivalence classes of $\sim$ which we recall forms a partition of A. What are the equivalence classes of $\overset{\mathcal{P}}{\equiv}$.

Rearrangements

Example

How many rearrangements of "AND" are there?

Example

How many rearrangements of "ALL" are there?

Solution

Let $L = \{w = (x, y, z) \ : \ x, y, z \in \{A, L_1, L_2\}\}$.
Note that $|L| = 3! = 6$.
Now we define a relation $\equiv$ on L as follows.
We say that $w_1 \equiv w_2$ iff when we erase the subscripts, the words become the same.
Note that $\equiv$ is an equivalence relation on L. (CHECK THIS!).
Now, count the equivalence classes any word $w \in L$.
There are 2 words in each equivalence class, and thus $\frac{6}{2} = 3$ rearrangements

Example

How many rearrangements of "MISSISSIPPI" are there?

Theorem

Let R be an equivalence relation on a finite set A. If all of the equivalence classes of R have the same size m, then the number of equivalence classes is given by

$$\#\text{ equiv classes} = \frac{|A|}{m}.$$

Note

Not all equivalence relations have the property that their equivalence classes all have the same size.

Example

Let $A = 2^{\{1,2,3\}}$ and let R be the "has the same size" relation. List all of the equivalence classes of R.