MTHSC 3190 Section 3.15

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Definition

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Example

$$
\begin{aligned}
A &= \{1,2,3,4\} \\
\mathcal{P}_1 &= \{\{1,3\},\{2,4\}\} \\
\mathcal{P}_2 &= \{\{1,2,3\},\{4\}\}
\end{aligned}
$$

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2. $\emptyset \notin \mathcal{P}$.
3. If $T, S \in \mathcal{P}$, then

$$T \cap S = \begin{cases} \emptyset, & \text{if } T \neq S, \\ T & \text{if } T = S. \end{cases}$$

4. $\bigcup_{S \in \mathcal{P}} S = A$.

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We can restate Cor 14.13 as follows. Let R be an equivalence relation on a set A. The equivalence classes of R form a partition of A.

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Definition

Let $\mathcal{P}$ be a partition of a set A. Define an equivalence relation $\overset{\mathcal{P}}{\equiv}$ on A as follows:

$$\overset{\mathcal{P}}{\equiv} \;= \{(x, y) \;:\; \exists S \in \mathcal{P} \text{ such that } x, y \in S\},$$

that is $x \overset{\mathcal{P}}{\equiv} y$ if and only if x and y are in the same part of $\mathcal{P}$.

Proposition

Let A be a set and let $\mathcal{P}$ be a partition of A. The relation $\overset{\mathcal{P}}{\equiv}$ is an equivalence relation on A.

Proof.

Reflexive

Symmetric

Transitive

$\square$

Proposition

Let $\mathcal{P}$ be a partition of a set A and let $\overset{\mathcal{P}}{\equiv}$ be defined as before. The equivalence classes of $\overset{\mathcal{P}}{\equiv}$. are precisely the parts of $\mathcal{P}$.

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Example

Let $A = 2^{\{1,2,3\}}$ and let $\sim$ be the "has the same size" relation. Let $\mathcal{P}$ be the set of equivalence classes of $\sim$ which we recall forms a partition of A. What are the equivalence classes of $\stackrel{\mathcal{P}}{\equiv}$.

Rearrangements

Example

How many rearrangements of "AND" are there?

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Let $L = \{w = (x, y, z) \;:\; x, y, z \in \{A, L_1, L_2\}\}$.

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Note that $|L| = 3! = 6$.

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Now, count the equivalence classes any word $w \in L$.
There are 2 words in each equivalence class, and thus $\frac{6}{2} = 3$ rearrangements.

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How many rearrangements of “MISSISSIPPI” are there?

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Theorem

Let R be an equivalence relation on a finite set A. If all of the equivalence classes of R have the same size m, then the number of equivalence classes is given by

$$\#\ \textit{equiv classes} = \frac{|A|}{m}.$$

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Example

Let $A = 2^{\{1,2,3\}}$ and let R be the "has the same size" relation. List all of the equivalence classes of R.