MTHSC 3190 Section 3.16

Kevin James

Definition

Let $n, k \in \mathbb{N}$. The symbol $\binom{n}{k}$ denotes the number of k-element subsets of an n-element set. The symbol is read as "n choose k."

Example

Compute $\binom{5}{0}$ and $\binom{5}{5}$.

Note

For $n \in \mathbb{N}$, $\binom{n}{n} = \binom{n}{0} = 1$.

Example

Compute $\binom{5}{1}$ and $\binom{5}{4}$.

Note

For $n \in \mathbb{N}$, $\binom{n}{1} = \binom{n}{n-1} = n$.

Proposition

Let $n, k \in \mathbb{N}$. with $0 \leq k \leq n$. Then

$$\binom{n}{k} = \binom{n}{n-k}.$$

Proof.

$\square$

Note

Thus we know that $\binom{5}{2} = \binom{5}{3}$ even though we do not know the value of either.

Example

Now, let's carefully compute $\binom{5}{2} = \binom{5}{3}$.

Lemma

Let $m \in \mathbb{N}$. Then $\sum_{j=1}^{m} j = \frac{m(m+1)}{2}$.

Proposition

Let $n \in \mathbb{N}$. Then $\binom{n}{2} = \binom{n}{n-2} = \frac{n(n-1)}{2}$.

Proof.

Note that there are there are $n - j$ subsets of size 2 whose smallest element is j, $(1 \leq j \leq n-1)$.

Thus the number of size 2 subsets is

$$\sum_{j=1}^{n-1}(n-j) = \sum_{k=1}^{n-1} k = \frac{n(n-1)}{2} \quad \text{from our lemma.}$$

$\square$

Example

1. Compute $\binom{n}{k}$ for $1 \leq n \leq 5$ and $0 \leq k \leq n$.
2. Compute $(x+y)^m$ for $1 \leq m \leq 5$.

Binomial Theorem

Let $n \in \mathbb{N}$. Then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Proof.

$\square$

Exercise

Arrange the binomial coefficients into a triangle. This is known as Pascal's tirangle. Write out the first few rows. Do you notice a relationship between consecutive rows?

Theorem

Let $n, k \in \mathbb{N}$. *Then* $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Combinatorial Proof.

$\square$

A formula for $\binom{n}{k}$

Theorem

Suppose that $n, k \in \mathbb{N}$. Then

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Proof.

$\square$