MTHSC 3190 Section 4.19

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Recall

The conditional statement $A \to B$ and its contrapositive $\neg B \to \neg A$ are logically equivalent.

Proof by Contrapositive

To prove "If A then B" prove instead the logically equivalent statement "If $\neg B$ then $\neg A$.

Proposition

Let m be an integer. If m^2 is even, then m is even.

Note

The challenge with proving the statement "If m^2 is even, then m is even" directly is that writing $m^2 = 2k$ isn't enough. For example, 6 is even, but it is not a square. A proof by the contrapositive makes the statement much easier to prove.

Proposition

Let x be an integer. If $x^2 + 2x < 0$, then $x < 0$.

Note

Consider the statement $A \rightarrow$ FALSE
Recalling our work with truth tables and boolean algebra, if this statement is true then what can we say about A?
A must be FALSE if the statement is true.

Proof by Contradiction

In order to prove a statement S, it is sufficient to prove $(\neg S \Rightarrow \text{FALSE})$ is a true statement.
That is we argue as follows.

Proof.

For the sake of contradiction, assume that $\neg S$ is true.

$$\vdots$$

Deduce a statement which is obviously false.
Then you have proved that $(\neg S \Rightarrow \text{FALSE})$ and thus $\neg S$ is false which means that S is true. □

Proposition

No integer is both even and odd.

Note

Note that this is equivalent to
$(\forall x \in \mathbb{Z}, \neg(x \text{ is even and } x \text{ is ood.}))$ which is equivalent to
$(\forall x \in \mathbb{Z}, x \text{ is not even OR } x \text{ is not odd})$.

Proof.

For the sake of contradiction, we will assume
$\neg(\forall x \in \mathbb{Z}, x \text{ is not even OR } x \text{ is not odd})$, which is equivalent to
$\exists x \in \mathbb{Z}, x$ is even AND x is odd.
Then $\exists k, m \in \mathbb{Z}$ such that $x = 2k$ and $x = 2m + 1$.
Thus, $2x = x + x = 2k + 2m + 1$.
which implies that $1 = 2(x - k - m)$.
Thus we have proved that 1 is an even number which is FALSE.
Thus our assumption must have been false so we have proved the

Proving If-Then statements by contradiction

1. $(A \to B) = (\neg A \vee B)$ Write out the truth table for this.
2. Thus to prove $A \Rightarrow B$ by contradiction, we prove $((A \wedge \neg B) \Rightarrow \text{FALSE})$.
3. Such a proof begins as follows. Assume for the sake of contradiction $(A \wedge \neg B)$ .... We then deduce a statement which is false.

Proposition

Suppose that $a, b \in \mathbb{Z}$. If $a \neq 0$, then there is at most one integer x such that $ax + b = 0$.