MTHSC 3190 Section 4.20

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Suppose that we want to prove a statement of the form $A \rightarrow B$.
We will proceed as in proof by contradiction. That is, we assume there is a counterexample.
Now, if the statement is a statement about natural numbers then there will be a *smallest counterexample*.
Use this idea of "smallest counterexample" to derive a false statement.
Then we have a proof of the statement by contradiction.

Proof by smallest counterexample

Suppose that we want to prove a statement of the form $A \to B$.

1. For the sake of contradiction, assume that the statement is false.
2. Let x be the smallest counterexample to the statement. *It must be clear that a "smallest counterexample" must exist in the case that the statement is false.*
3. (**Basis Step**) Check that $0 \notin X$.
4. (**Step Back**) Consider an instance of the result that is just smaller than x (typically $x - 1$ or $x/2$). Then the result must be true for this smaller value. Use this to deduce a contradiction. (*Many times, the contradiction is to show that the result must also hold for x which we assumed it did not.*)
5. Since we deduced a contradiction from our assumption that the statement was false, it now follows that the statement is true.

Proposition

Every natural number n is either even or odd.

Proof.

□

Corollary

Every integer is either even or odd.

Proof.

□

Proposition

Let $n \in \mathbb{N}$. The sum of the first n odd natural numbers is n^2. That is, $\sum_{k=1}^{n}(2k-1) = n^2$.

Proof.

□

Well Ordering Principle

Every nonempty set of natural numbers contains a smallest element.

Example

1. Let $XS = \{x \in \mathbb{N} \ : \ x \text{ is prime}\}$. The smallest element is ____________.
2. Let $Y = \{y \in \mathbb{Q} \ : \ y > 0\}$. Note that this set has no smallest element.
3. Let $X = \{x \in \mathbb{N} \ : \ x \text{ is both even and odd}\}$. This set also has no smallest element because it is empty.

Proof by Well-ordering principle

To prove a statement S about natural numbers:

1. Let X be the set of counterexamples to S. Make sure it is clear that $X \subseteq \mathbb{N}$.
2. (**Basis Step:**) Check that $0 \notin X$.
3. Suppose that $m \in X$.
4. (**Step Back:**) Pick a natural number $0 \leq y(m) < m$ (typically $y(m) = m - 1$). Show that $(m \in X \Rightarrow y \in X)$. Note that it may be easier to show the contrapositive $(y \notin X \Rightarrow m \notin X)$.
5. Note that you have proved that $X \subseteq \mathbb{N}$ has no smallest element and thus is empty.

Proposition

Let $a \neq 0, 1$. For $n \in \mathbb{N}$, $\sum_{k=0}^{n} a^k = \frac{a^{n+1}-1}{a-1}$.

Question

Which is bigger 2^n or n^2?

Example

Let's see....

n	2^n	n^2
0	1	0
1	2	1
2	4	4
3	8	9
4	16	16
5	32	25
6	64	36
7	128	49
8	256	64
9	512	81
10	1024	100

Proposition

For all $n \geq 5$, $2^n > n^2$.

Proof.

□

Definition (Recursive)

We define the Fibonacci sequence as follows.

$$F_0 = F_1 = 1; \qquad \text{For } n \geq 2, F_n = F_{n-1} + F_{n-2}.$$

Proposition

For all $n \in \mathbb{N}$, $F_n \leq 1.7^n$.