MTHSC 3190 Section 4.20

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Then we have a proof of the statement by contradiction.

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4. (**Step Back**) Consider an instance of the result that is just smaller than x (typically $x - 1$ or $x/2$). Then the result must be true for this smaller value. Use this to deduce a contradiction. (*Many times, the contradiction is to show that the result must also hold for x which we assumed it did not.*)

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5. Since we deduced a contradiction from our assumption that the statement was false, it now follows that the statement is true.

Basis Step

When proving $A \to B$. The basis step should ensure that your step-back value(s) will satisfy A. This is so that you may assume B for these values and argue to a contradiction. **So pay attention to the conditions in A and make sure that you check the correct instances of the claim to ensure that the step back works quickly.**

Proposition

Every natural number n is either even or odd.

Proof.

$\square$

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Corollary

Every integer is either even or odd.

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Let $n \in \mathbb{N}$. The sum of the first n odd natural numbers is n^2. That is, $\sum_{k=1}^{n}(2k-1) = n^2$.

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2. Let $Y = \{y \in \mathbb{Q} \ : \ y > 0\}$. Note that this set has no smallest element.
3. Let $X = \{x \in \mathbb{N} \ : \ x \text{ is both even and odd}\}$. This set also has no smallest element because it is empty.

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5. Note that you have proved that $X \subseteq \mathbb{N}$ has no smallest element and thus is empty.

Proposition

Let $a \neq 0, 1$. *For* $n \in \mathbb{N}$, $\sum_{k=0}^{n} a^k = \frac{a^{n+1}-1}{a-1}$.

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5	32	25
6	64	36
7	128	49
8	256	64
9	512	81
10	1024	100

Proposition

For all $n \geq 5$, $2^n > n^2$.

Proof.

$\square$

Definition (Recursive)

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For all $n \in \mathbb{N}$, $F_n \leq 1.7^n$.