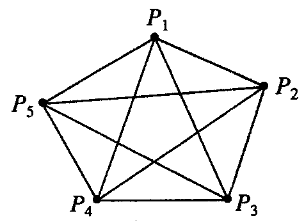
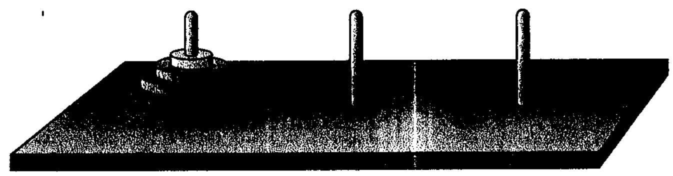
☆ **10.** Let $P_1, P_2, \ldots, P_n$ be n points in a plane with no three points collinear. Show that the number of line segments joining all pairs of points is $\frac{n^2 - n}{2}$. See the figure for $n = 5$.



☆ **11.** A puzzle called the Towers of Hanoi consists of a board with 3 pegs and several disks of differing diameters that fit over the pegs. In the starting position all the disks are placed on one peg, with the largest at the bottom, and the others with smaller and smaller diameters up to the top disk (see the figure). A move is made by lifting the top disk off a peg and placing it on another peg so that there is no smaller disk beneath it. The object of the puzzle is to transfer all the disks from one peg to another.



With a little practice, perhaps using coins of various sizes, you should convince yourself that if there are 3 disks, the puzzle can be solved in 7 moves. With 4 disks, 15 moves are required. Use the PMI to prove that with n disks, the puzzle can be solved in $2^n - 1$ moves. (*Hint:* In the inductive step you must *describe the moves* with $n + 1$ disks, and use the hypothesis of induction to count them.)

12. In a certain kind of tournament, every player plays every other player exactly once and either wins or loses. There are no ties. Define a *top* player to be a player who, for every other player x, either beats x or beats a player y who beats x.

(a) Show that there can be more than one top player.

(b) Use the PMI to show that every n-player tournament has a top player.

Proofs to Grade

13. Assign a grade of A (correct), C (partially correct), or F (failure) to each. Justify assignments of grades other than A.

★ **(a)** **Claim.** All horses have the same color.

"Proof." We must show that for all $n \in \mathbb{N}$, in every set of n horses, all horses in the set have the same color. Clearly in every set containing exactly 1 horse, all horses have the same color.

Now suppose all horses in every set of n horses have the same color. Consider a set of $n + 1$ horses. If we remove one horse, the horses in the remaining set of n horses all have the same color. Now consider a set of n horses obtained by removing some other horse. All horses in this set have the same color. Therefore all horses in the set of $n + 1$ horses have the same color. By the PMI, the statement is true for every $n \in \mathbb{N}$. ∎

★ **(b)** **Claim.** For all $n \in \mathbb{N}$, $n^3 + 44n$ is divisible by 3.

"Proof."

(i) $1^3 + 44(1) = 45$, which is divisible by 3, so the statement is true for $n = 1$.

(ii) Assume the statement is true for some $n \in \mathbb{N}$. Then $n^3 + 44n$ is divisible by 3. Therefore $(n+1)^3 + 44(n+1)$ is divisible by 3.

(iii) By the PMI, the statement is true for all $n \in \mathbb{N}$. ■

(c) **Claim.** For every natural number n, $n^2 + n$ is odd.

"Proof." The number $n = 1$ is odd. Suppose $n \in \mathbb{N}$ and $n^2 + n$ is odd. Then

$$\begin{aligned}(n+1)^2 + (n+1) &= n^2 + 2n + 1 + n + 1 \\ &= (n^2 + n) + (2n + 2)\end{aligned}$$

is the sum of an odd and an even number. Therefore, $(n+1)^2 + (n+1)$ is odd. By the PMI, the property that $n^2 + n$ is odd is true for all natural numbers n. ■

(d) **Claim.** For every natural number n, the matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$

"Proof." Let

$$S = \left\{ n \in \mathbb{N} : \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \right\}.$$

Clearly,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

so $1 \in S$. Assume that

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{n+1} = \begin{bmatrix} 1 & n+1 \\ 0 & 1 \end{bmatrix}.$$

Then $n + 1 \in S$, so $n \in S$ implies $n + 1 \in S$. By the PMI, $S = \mathbb{N}$. ■

★ **(e)** **Claim.** Every natural number greater than 1 has a prime factor.

"Proof."

(i) Let $n = 2$. Then n is prime.

(ii) Suppose k has a prime factor x. Then $k = xy$ for some y. Thus $k + 1 = xy + 1 = (x+1)(y+1)$, which is a prime factorization.

(iii) By the PMI, the theorem is proved. ■

(f) **Claim.** For all natural numbers $n \geq 4$, $2^n < n!$

"Proof." $2^4 = 16$ and $4! = 24$, so the statement is true for $n = 4$. Assume that $2^n < n!$ for some $n \in \mathbb{N}$. Then $2^{n+1} = 2(2^n) < 2(n!) \leq (n+1)(n!) = (n+1)!$, so $2^{n+1} < (n+1)!$. By the PMI, the statement is true for all $n \geq 4$. ■