§4.22 Recurrence Relations.

How does one discover formulas like

$$\sum_{i=0}^{n} i^2 = \frac{(2n+1)(n+1)n}{6} \quad \text{and}$$

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}} \quad ?$$

• First Order Recurrence Relations.

a sequence of integers $\{a_n\}_{n=0}^{\infty}$ whose definition is recursive and of the form

$$a_n = s a_{n-1} + t \qquad (s, t \in \mathbb{Z})$$

and where a_0 is given.

Examples: $a_n = 5 a_{n-1}$

$a_1 =$

$a_2 =$

$a_3 =$

$a_4 =$

$\vdots$

$a_k =$

Can you prove this?

<u>Example:</u> $a_n = a_{n-1} * 6$

$a_1 =$

$a_2 =$

$a_3 =$

$a_4 =$

$\vdots$

$a_k =$

Can you prove this?

- <u>Recall:</u> If $a \neq 0,1$ then $\sum_{k=0}^{n} a^k = \frac{a^{n+1}-1}{a-1}$,
- <u>Example:</u> $a_n = 3a_{n-1} + 5$

$a_1 =$

$a_2 =$

$a_3 =$

$a_4 =$

$\vdots$

$a_k =$

Prop 22.1:

All solutions to the 1^{st} order recurrence relation

$$a_n = s a_{n-1} + t$$

where $s \neq 1$ have the form

$$a_n = c_1 s^n + c_2$$

where

$$c_1 =$$

$$c_2 =$$

Pf:

$$a_1 = s a_0 + t$$

$$a_2 = s a_1 + t =$$

$$a_3 =$$

$$a_4 =$$

$$\vdots$$

We claim that $a_k =$

We will prove this by induction:

- **Example:** Find a solution for the recurrence relation

$$a_0 = 2$$
$$a_n = 2a_{n-1} + 3$$

- **Prop. 22.3** The solution to $a_n = a_{n-1} + t$ is

$$a_n = a_0 + nt$$

- **Pf:** Exercise.

- **2nd Order Recurrence relations.**

sequences of integers $\{a_n\}_{n=0}^{\infty}$ whose definition is recursive and has the form

$$a_n = s_1 a_{n-1} + s_2 a_{n-2}$$

- **Example:** Consider

$$a_0 = a_1 = 1$$

$$a_n = [redacted]\, a_{n-1} + 6 a_{n-2}$$

Observations:

1. If a_n is a solution then so is $c\, a_n$ for any c

2. If a_n and a_n' are solutions then so is $a_n + a_n'$

Now suppose that a_0 and a_1 are given.

Can we find c_1 and c_2 such that

$$a_n = c_1 2^n + c_2 3^n$$

gives a solution to

$$a_n = a_{n-1} + 6\, a_{n-2}$$

with the given initial conditions?

Prop. 22.4:

Let $s_1, s_2 \in \mathbb{Z}$ and let r be a root of $x^2 - s_1 x - s_2 = 0$. Then r^n is a solution to

$$a_n = s_1 a_{n-1} + s_2 a_{n-2}.$$

Pf:

Thm 22.5: Let $s_1, s_2 \in \mathbb{Z}$ and let r_1 & r_2 be roots of $x^2 - s_1 x - s_2 = 0$. If $r_1 \neq r_2$ then every solution to

$$a_n = s_1 a_{n-1} + s_2 a_{n-2}$$

is of the form

$$a_n = c_1 r_1^n + c_2 r_2^n$$

Example: Recall the Fibonacci sequence

$$F_0 = F_1 = 1 \;;\quad F_n = F_{n-1} + F_{n-1}$$

Solve this relation.

What if the roots are not distinct?

- Example: $a_0 = 2 \quad a_1 = 3$

  $a_n = 6a_{n-1} - 9a_{n-2}$

  Polynomial: $x^2 - 6x + 9 = (x-3)^2$

- Does: $a_n = c\,3^n$ work?

- New Idea:

- Thm 22.9: Let $s_1 \,\&\, s_2 \in \mathbb{Z}$ be s.t. $x^2 - s_1 x - s_2 = 0$ has only one root.

  Then every solution to

  $$a_n = s_1 a_{n-1} + s_2 a_{n-2}$$

  has the form

  $$a_n = c_1 r^n + c_2 n r^n$$