MTHSC 3190 Section 4.23

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Definition

A relation f is called a <u>function</u> provided that

$$(a, b), (a, c) \in f \Rightarrow b = c.$$

(-i.e. If $(a, b) \in f$, then $\forall c \neq b, (a, c) \notin f$.

Example

$f = \{(1,3),(2,5),(3,6),(4,7)\}$ is a function.
$Q = \{(1,2),(2,4),(1,3),(5,7)\}$ is not a function.

Notation

If f is a function and $(a, b) \in f$, we write $f(a) = b$.

Example

So, with f as in our previous example,
$f(1) = 3$
$f(2) =$ ________ $f(3) =$ ________ $f(4) =$ ________

Example

Express the integer function $f(x) = x^3$ as a set of ordered pairs.

Definition

The domain and image of a function f are defined as follows.

$$\begin{aligned}\mathrm{dom}(f) &= \{a : (a, *) \in f\}, \\ \mathrm{im}(f) &= \{b : (*, b) \in f\}.\end{aligned}$$

Example

Suppose that $f = \{(1, 2), (2, 4), (3, 5), (4, 7), (5, 10)\}$. What are the domain and image of f?

Notation

Suppose that f is a function and A and B are sets. We write

$$f : A \to B$$

to indicate that

$$\mathrm{dom}(f) = A, \qquad \text{and} \qquad \mathrm{im}(f) \subseteq B.$$

Showing f is a function

1. Check that f is a set of ordered pairs.
2. Suppose that $(a, b), (a, c) \in f$. Argue that $b = c$.

Showing $f : A \to B$

To show that $f : A \to B$,

1. Show that f is a function.
2. Show that $\mathrm{dom}(f) = A$ (using a set equality proof).
3. Show that $\mathrm{im}(f) \subseteq B$.

Example

Let $f = \{(x, x^2) \ : \ x \in \mathbb{Z}\}$. Show that $f : \mathbb{Z} \to \mathbb{N}$.

Proposition

Let A and B be finite sets. The number of functions $f : A \to B$ is given by $|B|^{|A|}$.

Notation

We denote the set of functions f from a set A to a set B by

$$B^A = \{f : A \to B\}.$$

Proof.

$\square$

Example

List all functions $f : \{1, 2, 3\} \to \{4, 5\}$.

Definition

A function is one-to-one (1-1) if whenever $(x, b), (y, b) \in f$, $x = y$. (-i.e. $x \neq y \Rightarrow f(x) \neq f(y)$.)

Proposition

Let f be a function. The inverse relation f^{-1} is a function if and only if f is 1-1.

Proof.

Exercise. $\square$

Proposition

Suppose that f and f^{-1} are functions. Then

$$\begin{aligned} dom(f) &= im(f^{-1}), \\ im(f) &= dom(f^{-1}. \end{aligned}$$

Proof.

Exercise. $\square$

Showing a function is 1-1

Direct Method Suppose that $f(x) = f(y)$. Then argue that $x = y$.

Contrapositive Suppose that $x \neq y$ then argue that $f(x) \neq f(y)$.

Contradiction Assume for the sake of contradiction that $f(x) = f(y)$ and $x \neq y$. Then argue to a contradiction.

Example

Let $f(x) = 3x + 4$. Show that f is 1-1.

Definition (Onto)

Let $f : A \to B$. We say that f is onto if $\mathrm{im}(f) = B$.

Note

We already know that $\mathrm{im}(f) \subseteq B$.

Proving Onto

To prove $f : A \to B$ is onto one may use either of the following methods.

Direct Method Let $b \in B$. Argue that there is $a \in A$ such that $f(a) = b$.

Subset Method Show that $B \subseteq \mathrm{im}(f)$.

Example

Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x + 7$. Show that f is onto.

Example

Let $f \mathbb{Q} \to \mathbb{Q}$ be defined by $f(x) = 5x + 2$. Is f 1-1? Is it onto?

Theorem

Suppose that $f : A \to B$. Then $f^{-1} : B \to A$ if and only if f is 1-1 and onto.

Proof.

$\square$

Definition

A function f is called a bijection if f is 1-1 and onto.

Example

$f(x) = x + 1$ is a bijection $\mathbb{Z} \to \mathbb{Z}$.

Pigeon hole principle—

Let A, B be finite sets and let $f : A \to B$.

1. f is 1-1 $\Rightarrow |A| \leq |B|$.
2. f is onto $\Rightarrow |A| \geq |B|$.

Note

Take a moment to write the contrapositives of the above statements.

Corollary

Let A, B be finite sets and let $f : A \to B$. If f is a bijection then $|A| = |B|$.

Proof.

$\square$

Theorem

Suppose that A and B are finite sets with $|A| = a$ and $|B| = b$. Then

1. $\#\{f : A \to B\} = b^a$.
2. $\#\{f : A \to B \mid f \text{ is 1-1}\} = \begin{cases} (b)_a & \text{if } a \leq b, \\ 0 & \text{otherwise.} \end{cases}$
3. $\#\{f : A \to B \mid f \text{ is onto}\} =$
$\begin{cases} \sum_{j=0}^{b} (-1)^j \binom{b}{j} (b-j)^a & \text{if } a \geq b, \\ 0 & \text{otherwise.} \end{cases}$
4. $\#\{f : A \to B \mid f \text{ is a bijection}\} = \begin{cases} a! & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$