MTHSC 3190 Section 4.23

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Definition

A relation f is called a function provided that

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Example

Express the integer function $f(x) = x^3$ as a set of ordered pairs.

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The domain and image of a function f are defined as follows.

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Notation

Suppose that f is a function and A and B are sets. We write

$$f : A \to B$$

to indicate that

$$\mathrm{dom}(f) = A, \qquad \text{and} \qquad \mathrm{im}(f) \subseteq B.$$

Showing f is a function

1. Check that f is a set of ordered pairs.
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1. Show that f is a function.
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Example

Let $f = \{(x, x^2) \ : \ x \in \mathbb{Z}\}$. Show that $f : \mathbb{Z} \to \mathbb{N}$.

Proposition

Let A and B be finite sets. The number of functions $f : A \to B$ is given by $|B|^{|A|}$.

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We denote the set of functions f from a set A to a set B by

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List all functions $f : \{1, 2, 3\} \to \{4, 5\}$.

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Proposition

Let f be a function. The inverse relation f^{-1} is a function if and only if f is 1-1.

Proof.

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Suppose that f and f^{-1} are functions. Then

$$\begin{aligned} dom(f) &= im(f^{-1}), \\ im(f) &= dom(f^{-1}). \end{aligned}$$

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Showing a function is 1-1

Direct Method Suppose that $f(x) = f(y)$. Then argue that $x = y$.

Contrapositive Suppose that $x \neq y$ then argue that $f(x) \neq f(y)$.

Contradiction Assume for the sake of contradiction that $f(x) = f(y)$ and $x \neq y$. Then argue to a contradiction.

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We already know that $\mathrm{im}(f) \subseteq B$.

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Proving Onto

To prove $f : A \to B$ is onto one may use either of the following methods.

Direct Method Let $b \in B$. Argue that there is $a \in A$ such that $f(a) = b$.

Subset Method Show that $B \subseteq \mathrm{im}(f)$.

Example

Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x + 7$. Show that f is onto.

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Theorem

Suppose that $f : A \to B$. Then $f^{-1} : B \to A$ if and only if f is 1-1 and onto.

Proof.

$\square$

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Pigeon hole principle—

Let A, B be finite sets and let $f : A \to B$.

1. f is 1-1 $\Rightarrow |A| \leq |B|$.
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Note

Take a moment to write the contrapositives of the above statements.

Corollary

Let A, B be finite sets and let $f : A \to B$. If f is a bijection then $|A| = |B|$.

Proof.

$\square$

Theorem

Suppose that A and B are finite sets with $|A| = a$ and $|B| = b$. Then

1. $\#\{f : A \to B\} = b^a$.
2. $\#\{f : A \to B \mid f \text{ is 1-1}\} = \begin{cases} (b)_a & \text{if } a \leq b, \\ 0 & \text{otherwise.} \end{cases}$
3. $\#\{f : A \to B \mid f \text{ is onto}\} =$
   $\begin{cases} \sum_{j=0}^{b} (-1)^j \binom{b}{j} (b-j)^a & \text{if } a \geq b, \\ 0 & \text{otherwise.} \end{cases}$
4. $\#\{f : A \to B \mid f \text{ is a bijection}\} = \begin{cases} a! & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$